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Keble College - Hilary 2015

CP3&4: Mathematical methods I&II
Tutorial 6 - Advanced problems

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 8th week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading and additional problem sets: all waves and vector calculus.

Goals

- Become more efficient at solving problems.
- Review the material covered this term.

Problems

Here we again get to work on our problem-solving abilities. Let's start with a review of vector calculus and then move on to waves.

1. The field $\mathbf{H}$ of a magnetic dipole of moment $\mathbf{M}$ placed at the origin is given by $\mathbf{H} = -\boldsymbol{\nabla}\Omega$, where Ω is the magnetostatic potential given by $\Omega = -\frac{1}{4\pi}\mathbf{M}\cdot\boldsymbol{\nabla}(\frac{1}{r})$. Find an expression for $\mathbf{H}$ in terms of $\mathbf{M}$ and $\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$.

2. State the vector condition for two surfaces $v(x, y, z) = 0$ and $w(x, y, z) = 0$ to cut orthogonally.

Prove that
$$\boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w = \frac{1}{2}\left(\nabla^2(vw) - v\nabla^2 w - w\nabla^2 v\right).$$

Hence or otherwise calculate all values of the constant a for which the surfaces $ay - x = 0$ and $5x^2 + y^2 + 3xy + z^2 = 25$ cut orthogonally. Sketch and describe a geometric interpretation for each of these values of a.

3. Given that $u = xy$ and $v = y/x$, show that $\partial(u, v)/\partial(x, y) = 2y/x$. Hence evaluate the integral
$$\iint_{\mathcal{S}} \mathrm{e}^{-xy}\,\mathrm{d}x\,\mathrm{d}y,$$
over the region $\mathcal{S}$ bounded by $x > 0$, $y > 0$, $xy < 1$, $1/2 < y/x < 2$.

4. With $\hat{\mathbf{n}}$ the unit normal to the surface S, evaluate $\int \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ for $\mathbf{F} = (y, 2x, -z)$ and S the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

Solution: 108.

5. O is the origin and A, B, C are points with position vectors $\mathbf{a} = (1, 0, 0)^{\mathrm{T}}$, $\mathbf{b} = (1, 1, 1)^{\mathrm{T}}$ and $\mathbf{c} = (0, 2, 0)^{\mathrm{T}}$, respectively. Find the vector area $\mathbf{S}$ of the loop $OABCO$

(i) by drawing the loop in projection onto the yz, zx and xy planes and calculating the components of $\mathbf{S}$, and

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. S. Rawlings and past Oxford Prelims exam questions.

(ii) by filling the loop with (e.g. 2 or 3) plane polygons, ascribing a vector area to each and taking the resultant.

Calculate the projected area of the loop (a) when seen from the direction which makes it appear as large as possible, and (b) when seen from the direction of the vector $(0,-1,1)^{\mathrm{T}}$? What are the corresponding answers for the loop $OACBO$?

Solution: $OABCO$: $\mathbf{S}=(-1,-1/2,3/2)^{\mathrm{T}}$, $\sqrt{14}/2$, $\sqrt{2}$; $OACBO$: $\mathbf{S}=(1,0,0)^{\mathrm{T}}$, 1, 0.

6. Calculate the line integral

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

where $\mathbf{F}=y\mathbf{i}-x\mathbf{j}+\mathbf{k}$, in the *clockwise* direction around the curve $x^2+y^2=4$. Verify your answer using Stokes' Theorem.

7. (a) Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2}-c^2\frac{\partial^2 y(x,t)}{\partial x^2}=0\,,$$

describing small amplitude transverse waves on a uniform string of mass per unit length ρ and tension T. Derive also a formula for the wave speed c. You may assume that $\partial y/\partial x$ is small and that gravity can be neglected. [Since we have covered it in a previous tutorial, omit this part of the question if you are completely comfortable with it.]

(b) A string is held taught between two fixed points at $x=0$ and $x=l$ and made to vibrate. Assuming that no energy is lost as the string vibrates, show that standing waves are set up that satisfy the time-independent form of the wave equation

$$\frac{\partial^2 y}{\partial x^2}+k^2 y=0\,,$$

and derive an expression for the set of possible frequencies of the standing waves.

(c) At time $t=0$ the string is stationary and held in the form

$$y(x)=A\sin\left(\frac{\pi x}{l}\right)\left(1+\cos\left(\frac{\pi x}{l}\right)\right)\,,$$

where A is a constant. Find the amplitudes of the normal modes that exist on the string after it is released.

(d) Derive expressions for the kinetic and potential energy of each normal mode on a vibrating string. Use your expressions to calculate the total energy of each mode from part (c), in terms of A and the properties of the string.

8. The propagation of transverse waves down a stretched string of tension T and density ρ is described by the wave equation

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\,,$$

where y is the transverse displacement at point x at time t and $v=\sqrt{\frac{T}{\rho}}$ is the speed of propagation.

A long string, held under tension T, is made of two pieces which are joined at $x=0$. The densities of the string are ρ_1 for $x<0$ and ρ_2 for $x>0$. A sinusoidal wave $y=A\exp\{\mathrm{i}(\omega t-k_1 x)\}$ travels from $x<0$ towards the boundary. Find the amplitudes of the reflected and the transmitted waves in terms of ρ_1, ρ_2 and A.

A mass m is now placed at the join of the string at $x=0$. Show that the ratio of the transmitted and incident amplitudes is now

$$\left|\frac{A_t}{A}\right|=\frac{2\sqrt{\rho_1}}{\left[\left(\sqrt{\rho_1}+\sqrt{\rho_2}\right)^2+\alpha^2\right]^{\frac{1}{2}}}$$

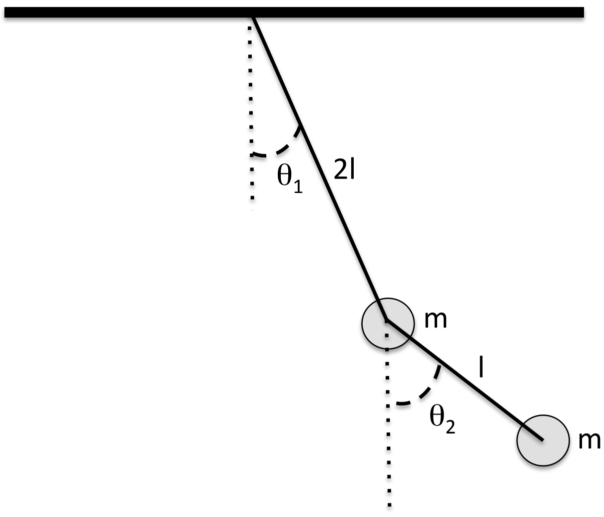
and find an expression for α. What is the phase of the transmitted wave with respect to the incident wave?

9. The properties of a string are altered so that the wave equation describing small amplitude transverse waves on the string becomes

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = -\mu^2 y(x,t)\,.$$

By utilizing the *ansatz*, $y(x,t) = \Re[\exp i(\omega t \pm kx)]$, or otherwise, find the relation that the modified wave equation implies between the wavenumber k and angular frequency ω for a string of infinite extent. Compute the phase velocity v_p and group velocity v_g of the waves as a function of wavenumber, and comment on the relation of these to c. What are the limiting behaviours of both v_p and v_g as $k \to 0$ and $k \to \infty$?

10. Consider a double pendulum, comprising of two identical masses m, suspended using two massless rods. The first rod, of length $2l$, is attached to a fixed pivot point in the ceiling, whilst the second rod, of length l, is attached to the first mass, allowing frictionless rotation of the system in the vertical plane (see figure below).



(a) Taking the zero of potential energy to be at the equilibrium position and measuring from the pivot point, derive an expression for the potential energy of the system, U, in terms of the respective angles θ_1 and θ_2 between the two rods and the vertical direction.

(b) Derive an expression for the kinetic energy of the system, K, in terms of θ_1 and θ_2.

(c) Assuming that the angles θ_1 and θ_2 are small, simplify the expressions for U and K obtained in (a) and (b). Use the vector notation

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

to rewrite these expressions as $U = \frac{1}{2}\boldsymbol{\theta}^{\mathbf{T}}\mathbf{V}\boldsymbol{\theta}$ and $K = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathbf{T}}\mathbf{T}\dot{\boldsymbol{\theta}}$. Derive explicit expressions for the matrices $\mathbf{V}$ and $\mathbf{T}$.

(d) Use the conservation of energy to derive the equations of motion of the system in matrix form.

(e) Find the normal modes of oscillation of the system.

Class problems

11. Find $\nabla\phi$ in the cases: (a) $\phi = \ln|\mathbf{r}|$; (b) $\phi = r^{-1}$, where $r = |\mathbf{r}|$.

12. Verify the relation

$$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) - \mathbf{F} \times (\nabla\phi),$$

where ϕ is a scalar field and $\mathbf{F}$ is a vector field.

13. A solid hemisphere of uniform density k occupies the volume $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$. Using symmetry arguments wherever possible, find

(i) its total mass M,

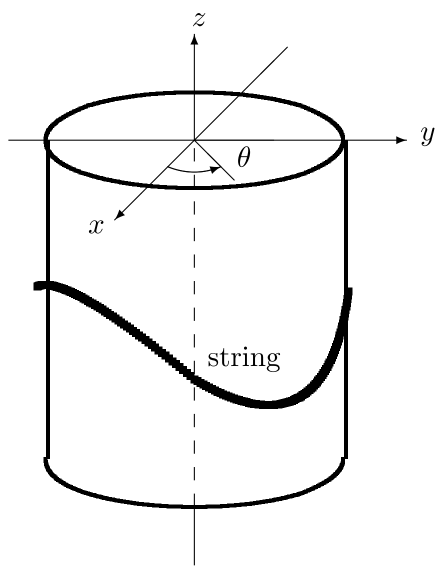
(ii) the position $(\bar{x}, \bar{y}, \bar{z})$ of its centre-of-mass, and

(iii) its moments and products of inertia, I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}, where

$$I_{zz} = \int k(x^2 + y^2)\,\mathrm{d}V\,, \qquad I_{xy} = \int k\,xy\,\mathrm{d}V\,, \qquad \text{etc.}$$

Solution: $M = 2\pi a^3 k/3$, $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3a/8)$, $I_{xx} = I_{yy} = I_{zz} = 4\pi k a^5/15$, $I_{xy} = I_{yz} = I_{zx} = 0$.

14. A circular closed string of linear mass density ρ is held under tension T on a smooth cylinder of radius R as shown in the figure. The string can undergo small transverse vibrations on the cylinder surface in the z-direction. Any effects caused by the curvature of the cylinder may be neglected. Write down the wave equation for transverse vibrations of displacement $z(\theta, t)$ with $0 \leq \theta \leq 2\pi$ denoting the polar angle and t the time. State the boundary conditions obeyed by the string displacement z at $\theta = 0$ and $\theta = 2\pi$.



For which values of k and $\omega > 0$ is the complex function $y(\theta, t) = A\exp[\mathrm{i}(k\theta - \omega t)]$, with A the constant complex amplitude, a solution of the wave equation which obeys the boundary conditions? Give a physical interpretation of ω and k. Write down the two smallest allowed values ω_1 and $\omega_2 > \omega_1$ and the associated values of k explicitly. Find a superposition of solutions with $\omega = \omega_2$ for which $y(\pi/3, t) = 0$ at all times. At which other values of θ will the string not be displaced at any time?

Is $y(\theta) = A$ for $\omega = 0$ and $k = 0$ a valid solution? Find the general solution with $\partial^2 y/\partial t^2 = 0$. What kind of motion does this solution describe?

15. The angular frequency of waves in a dispersive medium is given by

$$\omega(k) = \sqrt{gk\tanh(kh)}$$

where k is the wave number and g and h are constants. Find expressions for the phase velocity v_p and the group velocity v_g. Simplify these expressions for v_p and v_g in the limiting cases $h \gg 1/k$ and $h \ll 1/k$.

By considering the superposition of two waves of different wavelength explain how wave packets will disperse in these two limiting cases, respectively.

Calculate the time that the peak of a wave packet with mean frequency 0.2Hz needs to travel a distance of 1000km for $h = 1\text{km}$ and $g = 10\text{m/s}^2$.

16. Two identical masses $m_1 = m_2 = m$ are connected by a massless spring with spring constant k. Mass m_1 is attached to a support by another massless spring with spring constant $2k$. The masses and springs lie along the horizontal x-axis on a smooth surface. The masses and the support are allowed to move along the x-axis only. The displacement of the support in the x-direction at time t is given by $f(t)$ and is externally controlled. Write down a system of differential equations describing the evolution of the displacements x_1 and x_2 of the masses from their equilibrium positions.

Determine the frequencies of the normal modes and their amplitude ratios.

The displacement of the support is given by $f(t) = A\sin(\omega t)$ with $\omega^2 = k/m$ and constant amplitude A. Find expressions for $x_1(t)$ and $x_2(t)$ assuming that any transients have been damped out by a small, otherwise negligible, damping term.