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Keble College - Hilary 2014
CP3&4: Mathematical methods I&II
Tutorial 5 - Waves and normal modes II

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 6th week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: DJM 2, RHB 9 and FR 5. Additional problem sets: NWI, NWII and NWIII.

Goals

- Understand normal modes as the basic building blocks for describing wave-like motion in discrete coupled degrees of freedom.
- Learn how to find the normal modes of linearly coupled oscillating systems, described by linearly couple ODEs.
- Know how to use the normal modes to solve for the motion of the system.

Problems

As described in the notes provided with this problem set, normal modes appear in the extension of waves to discrete degrees of freedom. They are the basic modes of vibrations. They naturally arise when discrete degrees of freedom obey linear coupled differential equations. Changing basis, some combinations of these degrees of freedoms, called the normal coordinates, obey uncoupled differential equations, and thus separately undergo simple harmonic motion. A normal mode is the mode of vibration where only one of the normal coordinates is oscillating with non-zero amplitude. The general solution is thus a superposition of the normal modes.

The notes discuss a general procedure for finding the normal modes. But the focus of this course will be on two degrees of freedoms. Let's start by considering a few examples and seeing what the resulting motion looks like.

1. Two simple pendula with positions x and y are of equal length l, but their bobs have different masses m_1 and m_2. They are coupled via a spring of spring constant k.

(a) Show that their equations of motion are

$$\ddot{x} = -\frac{g}{l}x - \frac{k}{m_1}(x-y) \quad \text{and} \quad \ddot{y} = -\frac{g}{l}y - \frac{k}{m_2}(y-x)\,.$$

(b) By taking suitable linear combinations of the two equations of motion, obtain two uncoupled differential equations for linear combinations of x and y. Hence find the normal mode frequencies and the relative amplitudes corresponding to each frequency. [Hint: These linear combinations are the normal coordinates. One of them is fairly obvious. For the other, it may be helpful to consider the centre of mass of the two bobs.]
Solution: $\omega_1^2 = g/l$, $\omega_2^2 = g/l + k/m_1 + k/m_2$, $\mathbf{v}_1 = (1,1)^{\mathrm{T}}$, $\mathbf{v}_2 = (1, -m_1/m_2)^{\mathrm{T}}$.

[1]These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. G.G. Ross and past Oxford Prelims exam questions.

(c) A more systematic way of doing this is with the matrix method. Write the equations as $\ddot{\mathbf{r}} + \mathbf{A}\mathbf{r} = 0$ with $\mathbf{r} = (x, y)^T$. Look for non-trivial solutions of the type $\mathbf{r} = \mathbf{v}_i e^{-i\omega_i t}$. The ω_i^2 are found from $| - \omega_i^2 \mathbf{1} + \mathbf{A}| = 0$. Then, for each ω_i, the amplitude $\mathbf{v}_i$ is found from $(-\omega_i^2 \mathbf{1} + \mathbf{A})\mathbf{v}_i = 0$. Make sure you understand why the method works.
Solution: $\omega_1^2 = g/l$, $\omega_2^2 = g/l + k/m_1 + k/m_2$, $\mathbf{v}_1 = (1, 1)^\mathrm{T}$, $\mathbf{v}_2 = (1, -m_1/m_2)^\mathrm{T}$.

(d) Find the most general solutions for x and y as a function of time t.
Solution: $(x, y)^\mathrm{T} = \sum_i \mathbf{v}_i c_i e^{-i\omega_i t} = \mathbf{v}_1(A \sin(\omega_1 t) + B \cos(\omega_1 t)) + \mathbf{v}_2(C \sin(\omega_2 t) + D \cos(\omega_2 t))$, where the arbitrary constants c_i are complex.

(e) At $t = 0$, both pendula are at rest, with $x = 0$ and $y = Y_0$. They are then released. Determine the subsequent motion of the system.
Solution: $(x, y)^\mathrm{T} = Y_0(\mathbf{v}_1 \cos(\omega_1 t) - \mathbf{v}_2 \cos(\omega_2 t))/(1 + m_1/m_2)$.

(f) At $t = 0$, both bobs are at their equilibrium positions. The first is stationary but the second is given an initial velocity v_0. Determine the subsequent motion of the system.

(g) Give two sets of initial conditions such that the subsequent motion of the pendula corresponds to each of the normal modes.

We now specialize to the case $m_1 = m_2 = m$ and $k/m = \alpha g/l$ with $\alpha \ll 1$.

(h) For the original initial conditions of part (d), show that

$$y(t) = D \cos(\Delta t) \cos(\bar{\omega} t), \quad \text{and} \quad x(t) = D \sin(\Delta t) \sin(\bar{\omega} t),$$

and determine D, Δ and $\bar{\omega}$.

(i) Sketch $x(t)$ and $y(t)$ for $\alpha = 0.105$, and note that the oscillations are transferred from the first pendulum to the second and back. Approximately how many oscillations does the second pendulum have before the first pendulum is oscillating again with its initial amplitude?

2. $*$ Two masses m are fixed to springs at B and C as shown in Fig. 1. They are displaced by small distances x_1 and x_2 from their equilibrium positions along the line of the springs, and execute small oscillations. The springs have negligible masses.

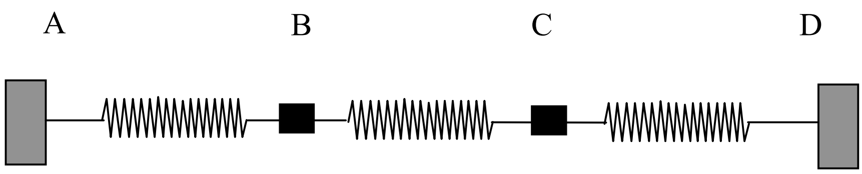

Figure 1: Masses connected by springs.

(a) AB, BC, and CD are identical springs with stiffness constant k. Show that the angular frequencies of the normal modes are

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \omega_3 = \sqrt{\frac{3k}{m}}.$$

Sketch how the two masses move in each mode. Find x_1 and x_2 at times $t > 0$ if at $t = 0$ the system is at rest with $x_1 = a$, $x_2 = 0$.

(b) Now consider the case where the springs AB and CD have stiffness constant k_0, while BC has stiffness constant k_1. If C is clamped, B vibrates with frequency $\nu_0 = 1.81$Hz. The frequency of the lower frequency normal mode is $\nu_1 = 1.14$Hz. Calculate the frequency of the higher frequency normal mode, and the ratio k_1/k_0.

An important concept in normal modes are the normal coordinates Q_i. These are the uncoupled degrees of freedom that undergo simple harmonic motion $Q_i = c_i e^{-i\omega_i t}$ at the normal mode frequencies ω_i. The approach of question **1***(b) involves starting by finding the normal coordinates. However, the more systematic method of* **1***(c) proceeds without finding the normal coordinates, though it is possible to find them by inverting the relationship $\mathbf{r} = \sum_i \mathbf{v}_i Q_i$.*

Even though we can solve for the evolution of the system without direct reference to them, normal coordinates are not irrelevant . They are the natural coordinates with which to describe the system. The physical problem is much more naturally expressed in terms of them, in the same way that choosing spherical coordinates for a problem with spherical symmetry is a sensible thing to do. We might even go as far to try and express the system in terms of uncoupled particles with displacements given by the normal coordinates, rather than the original coupled particles. The next problem explores this.

3. * The displacements of two coupled oscillators q_1 and q_2 with mass m are described by the coupled equations

$$m\ddot{q}_1 = k(q_2 - 2q_1), \quad \text{and} \quad m\ddot{q}_2 = k(q_1 - 2q_2),$$

where k is a constant. The normal coordinates are given by $Q_1 = (q_1 + q_2)/\sqrt{2}$ and $Q_2 = (q_1 - q_2)/\sqrt{2}$, which evolve according to

$$Q_1 = A\cos\omega_1 t + B\sin\omega_1 t, \quad \text{and} \quad Q_2 = C\cos\omega_2 t + D\sin\omega_2 t,$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ are the normal mode frequencies.

(a) Find expressions for $q_1(t)$ and $q_2(t)$.

(b) Since they are conservative, the forces appearing in the equations for q_1 and q_2 may be interpreted in terms of a potential energy function $V(q_1, q_2)$, as follows. We write the equations as

$$m\ddot{q}_1 = -\frac{\partial V}{\partial q_1}, \quad \text{and} \quad m\ddot{q}_2 = -\frac{\partial V}{\partial q_2}, \quad (\text{equivalently} \quad m\ddot{\mathbf{q}} = -\boldsymbol{\nabla} V,)$$

generalising the familiar one-dimensional equation $m\ddot{q} = \mathrm{d}V/\mathrm{d}q$. Show that V may be taken to be $V = k(q_1^2 + q_2^2 - q_1 q_2)$.

(c) Rewrite V in terms of the normal mode coordinates. Show that $m\ddot{\mathbf{Q}} = -\boldsymbol{\nabla} V$, where $\boldsymbol{\nabla}$ defined with reference to the normal coordinates.
Solution: $V = m\omega_1^2 Q_1^2/2 + m\omega_2^2 Q_2^2/2$.

(d) Calculate the total kinetic energy K of the two masses in terms of the normal coordinates.
Solution: $K = m(\dot{Q}_1^2 + \dot{Q}_2^2)/2$.

(e) Write down the total energy $V + K$ in terms of coordinates Q_1 and Q_2 and also in terms of q_1 and q_2. Discuss similarities and differences of these two expressions.

Sticking with two degrees of freedom there are still a few more possibilities to consider. The first is the existence of a zero-frequency mode.

4. Two particles 1 and 2, each of mass m, are connected by a light spring of stiffness k and are free to slide along a smooth long horizontal track. What are the normal frequencies of this system? Describe the motion in the mode of zero frequency. Why does a zero-frequency mode appear in this problem, but not in problem **2**, for example?

The other slightly more complicated thing is to consider driving, or external forces acting on the system. The next problem considers a constant (zero-frequency) force and the one after an oscillatory force, where we would expect to encounter resonance.

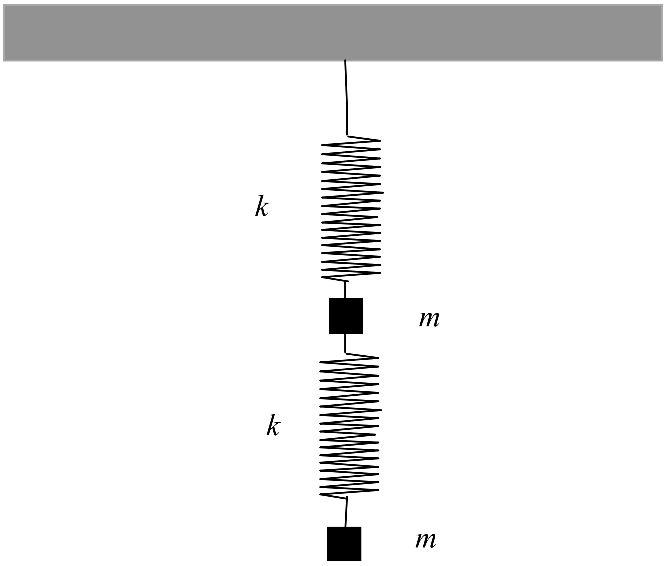

Figure 2: Masses connected by springs.

5. * We consider the setup in Fig. 2.

   Two equal masses m are connected as shown with two identical massless springs, of spring constant k. Considering only motion in the vertical direction, obtain the differential equations for the displacements of the two masses from their equilibrium positions. Show that the angular frequencies of the normal modes are given by

$$\omega_{1,2}^2 = \frac{(3 \pm \sqrt{5})k}{2m}.$$

   Find the ratio of the amplitudes of the two masses in each separate mode. Why does the acceleration due to gravity not appear in these answers?

6. Particle 1 in question **4** is now subject to a harmonic driving force $F \cos \omega t$. In the steady state, the amplitudes of vibration of 1 and 2 are A and B, respectively. Find A and B, and discuss qualitatively the behaviour of the system as ω^2 is slowly increased from values near zero to values greater than $2k/m$.

Class problems

7. The currents I_1 and I_2 in two coupled LC circuits satisfy the equations

$$L\frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} + \frac{I_1}{C} - M\frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} = 0, \quad \text{and} \quad L\frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} + \frac{I_2}{C} - M\frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} = 0,$$

   where $0 < M < L$. Find formulae for the two possible frequencies at which the coupled system may oscillate sinusoidally.

8. Solve the differential equations

$$2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 3y + z = \mathrm{e}^{2x}, \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} + 2y - z = 0,$$

   Is it possible to have a solution to these equations for which $y = z = 0$ when $x = 0$?

9. We extend problem **2** to n masses connected by identical springs. A Mathematica program to solve this problem is given in Fig. 3. Discuss the solutions for frequencies and amplitude ratios produced by this program.

```
In[1]:= A[n_] := DiagonalMatrix[Table[-2, {n}]] +
          DiagonalMatrix[Table[1, {n - 1}], 1] + DiagonalMatrix[Table[1, {n - 1}], -1]
```

Solve for n masses connected by springs

```
In[2]:= n = 100; vv = Eigensystem[N[A[n]]];
        omegas = Re[-I Sqrt[vv[[1]]]]; amplitudes = vv[[2]];
```

Consider the l-th normal mode

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In[3]:= l = 100; Print[l, "-th mode frequency is: ", omegas[[l]]]
        ListPlot[amplitudes[[l]], AxesLabel → {"mass position", "amplitude ratios"}]
```

Figure 3: Mathematica program for masses connected by springs.