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Keble College - Hilary 2015

CP3&4: Mathematical methods I&II

Tutorial 4 - Vector calculus and multiple integrals II

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 5th week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB 6, 10 and 11. Additional problem sets: MV.

Goals

- Use Stoke's and the divergence theorem to relate surface and volume integrals, and understand the physical significance of this.
- Learn how to use basis vectors relating to non-Cartesian coordinates.
- Become increasingly proficient at evaluating vector calculus integrals.

Problems

In the previous tutorial on vector calculus we introduced the objects of vector calculus, the scalar and vector fields, their gradients, div, grad and curl, their integration over lines, surfaces and volumes, and how these integrals may be re-expressed as multiple integrals. The end goal of this tutorial will be to evaluate complicated integrals of this type, particularly integrals over curved surfaces. Most of this will be achieved by learning how to reduce a complicated surface integral to a simpler integral. We have already seen examples of using symmetry arguments to reduce the number of calculations needed. We have also performed an appropriate parameterisation of the region of integration to solve the problem. And we have, for the case of line integrals, seen that on occasion one can deduce when a integral depends on the curve or integration or merely its initial and final points. We will add to these tools. We will learn to relate different types of integral to each other and see more examples of when the region of integration only depends on its boundaries. We will think about not only parameterising space with non-Cartesian coordinates but also working with components of the vectors in the corresponding non-Cartesian bases. After all this is finished, we'll be pretty well qualified students of vector calculus, and all we'll need is a bit of practice before we are comfortable with pretty much anything thrown at us.

The two most important theorems that allow us to relate integrals are Stoke's theorem

$$\oint_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \iint_{\mathcal{S}} \boldsymbol{\nabla} \times \mathbf{v} \cdot \mathrm{d}\mathbf{S},$$

where $\mathcal{C}$ is the loop on the boundary of surface $\mathcal{S}$, and the divergence (Gauss') theorem

$$\oiint_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{v} \mathrm{d}V,$$

where $\mathcal{S}$ is the closed surface enclosing volume $\mathcal{V}$. There are other useful or related theorems out there (see class problems), but these two are the most common examples.

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. S. Rawlings and past Oxford Prelims exam questions.

1. Briefly outline the proofs of Stoke's theorem and the divergence theorem. Also state them in differential form.

2. $*$ Verify Stokes' theorem for $\mathbf{v} = (y, -x, z)$ and the hemispherical surface $|\mathbf{r}| = 1$, $z \geq 0$.

3. $*$ The vector $\mathbf{F}$ is a function of position $\mathbf{r} = (x, y, z)$ and has components (xy^2, x^2, yz). Calculate the surface integral $\iint \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = 1$. Show that the integral $\iint \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ taken outwards over the whole surface is not zero. Show that it equals $\iiint \nabla \cdot \mathbf{F} \,\mathrm{d}V$ calculated over the volume of the prism. Explain this.

Stoke's and the divergence theorem tell us how to turn a line integral in to a surface integral, or a surface integral into a line integral and vice versa. As a consequence of this it also tells us when we can change our region of integration with no consequence. For example, consider a vector field $\mathbf{v}$ *with zero curl* $\boldsymbol{\nabla} \times \mathbf{v} = 0$. *It follows from Stoke's theorem that* $\oint_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$ *for all closed loops* $\mathcal{C}$. *The implication of this is that any two integrals* $\int_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\mathbf{l}$ *where* $\mathcal{C}$ *share the same boundary points must have the same value (so that the total of going around a loop formed by the two paths is zero). Similarly consider a vector field* $\mathbf{v}$ *with zero divergence* $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. *It follows from divergence theorem that* $\oiint_{\mathcal{S}} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = 0$ *for all closed surfaces* $\mathcal{S}$. *The implication of this is that any two integrals* $\iint_{\mathcal{S}} \mathbf{v} \cdot \mathrm{d}\mathbf{S}$ *where* $\mathcal{S}$ *share the same boundary lines must have the same value (so that the total of going over the close surface formed by the two paths is zero).*

Both of these facts imply the existence of some conserved quantities. And you can thus connect these to physical conserved quantities to help provide intuition for the maths.

4. Provide physical interpretations of the Stoke's theorem and divergence theorem. For the former, connect this to line integrals featuring exact differentials, conservative fields and their corresponding potentials, seen in the previous vector calculus tutorial. For the latter, connect this to flow into and out of a surface, and mass conservation or creation.

The following questions should be easier (and well-defined) now you understand Stoke's theorem and the divergence theorem.

5. Let $\mathbf{v} = (y, -x, 0)^T$. Find $\oint_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\mathbf{l}$ for a closed loop on the surface of the cylinder $(x-3)^2 + y^2 = 2$.

Solution: $0, -4\pi, \cdots$, depends on winding.

6. Air is flowing with a speed $0.4\mathrm{ms}^{-1}$ in the direction of the vector $(-1, -1, 1)$. Calculate the volume of air flowing per second through a loop that consists of straight lines joining, in turn, the following points (in units of meters): $(1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 0)$, $(0, 1, 1)$, $(1, 1, 1)$ and $(1, 1, 0)$.

Solution: $V = -\sqrt{3}/5\mathrm{m}^3\mathrm{s}^{-1}$.

7. $*$ Calculate the surface integral

$$\int_{\mathcal{S}} (3xz\hat{\mathbf{i}} + 2y\hat{\mathbf{j}}) \cdot \mathrm{d}\mathbf{S}\,,$$

where the surface is a hemisphere defined by the curve $x^2 + y^2 + z^2 = 4$ and $z \geq 0$.

We began our foray into vector calculus by taking the vectors $|v\rangle$ and $|r\rangle$ and representing them in a Cartesian basis so that we can entirely deal with column vectors $\mathbf{v} = (v_x, v_y, v_z)^T$ and $\mathbf{r} = (x, y, z)^T$. This suggests a decomposition of the vector fields $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ and vector differentials $\mathrm{d}\mathbf{l} = \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}} + \mathrm{d}z\hat{\mathbf{k}}$ in terms of basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. Since we know the actions of these basis vectors with the scalar and vector products $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = 0$, $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = 1$, $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{i}}\times\hat{\mathbf{i}} = 0$, and cyclic permutations, we know how to calculate quickly the scalar and vector products of vectors decomposed in terms of them, i.e.

$$\mathbf{u}\cdot\mathbf{v} = u_x v_x + u_y v_y + u_y v_y, \quad \textit{and} \quad \mathbf{u}\times\mathbf{v} = (u_y v_z - u_z v_y)\,\hat{\mathbf{i}} + (u_z v_x - u_x v_z)\,\hat{\mathbf{j}} + (u_x v_y - u_y v_x)\,\hat{\mathbf{k}},$$

This has certainly been instrumental to the success of what we have done so far.

What we realised in the last problem set is that for some choice of lines or surfaces, it was better to parameterise them in non-Cartesian coordinates. We found expressions for the line and surface elements in terms of corresponding basis vectors e.g. $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\rho}}$, and $\hat{\mathbf{r}}$. In systems with circular, cylindrical or spherical symmetry, we expect many of the vector fields to be also more simply expressed in such basis vectors. So here we look to go fully into the world of non-Cartesian coordinates. Not only do we parameterise our integration region according to them but we deal with the components of the vectors relating to the corresponding non-Cartesian basis vectors. Let us consider how such basis vectors are obtained.

We know that our position vector $\mathbf{r}(u, v, w)$ can be parameterised by some (u, v, w), which we call coordinates. These could be (x, y, z) or some other set of coordinates e.g. spherical (r, θ, ϕ). The vectors $\mathbf{e}_u = \frac{\partial\mathbf{r}}{\partial u}$, $\mathbf{e}_v = \frac{\partial\mathbf{r}}{\partial v}$ and $\mathbf{e}_w = \frac{\partial\mathbf{r}}{\partial w}$ tell us how $\mathbf{r}$ changes with each coordinate u, v and w. Normalising these vectors gives us basis vectors $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$ and $\hat{\mathbf{e}}_w$. These three basis vectors lie along the tangents at $\mathbf{r}$ to curves containing $\mathbf{r}$ in which only one coordinate is varied. For $(u, v, w) = (x, y, z)$ these curves are straight and parallel and their tangents are always in the same direction. Thus $\hat{\mathbf{e}}_x = \hat{\mathbf{i}}$, $\hat{\mathbf{e}}_y = \hat{\mathbf{j}}$ and $\hat{\mathbf{e}}_z = \hat{\mathbf{k}}$ are independent of position. In general, these curves of constant coordinates will be curved and, if straight, might not always be parallel. Hence the term curvilinear coordinates. A good example are spherical coordinates, whose lines of constant coordinates are curved or non-parallel (visualise it) and thus basis vectors $\hat{\mathbf{e}}_r = \hat{\mathbf{r}}$, $\hat{\mathbf{e}}_\theta = \hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{e}}_\phi = \hat{\boldsymbol{\phi}}$ are position dependent.

This position-dependence does not preclude the basis vectors from being useful. It may still be the case that at every $\mathbf{r}$ the basis vectors still obey the nice properties $\hat{\mathbf{e}}_u\cdot\hat{\mathbf{e}}_v = 0$, $\hat{\mathbf{e}}_u\cdot\hat{\mathbf{e}}_u = 1$, $\hat{\mathbf{e}}_u\times\hat{\mathbf{e}}_v = \hat{\mathbf{e}}_w$, $\hat{\mathbf{e}}_u\times\hat{\mathbf{e}}_u = 0$, and cyclic permutations. This occurs when the lines of constant coordinates always intersect at right angles in a way that forms a right-handed system. This is true for (x, y, z), (r, θ, ϕ) and (ρ, ϕ, z), and we only deal with curvilinear coordinates of this type. Thus using a basis of this type, we are once again able to decompose $\mathbf{v} = v_u\hat{\mathbf{e}}_u + v_v\hat{\mathbf{e}}_v + v_w\hat{\mathbf{e}}_w$ and obtain simple expressions

$$\mathbf{u}\cdot\mathbf{v} = u_u v_u + u_v v_v + u_w v_w, \quad \textit{and} \quad \mathbf{u}\times\mathbf{v} = (u_v v_w - u_w v_v)\,\hat{\mathbf{e}}_u + (u_w v_u - u_u v_w)\,\hat{\mathbf{e}}_v + (u_u v_v - u_v v_u)\,\hat{\mathbf{e}}_w.$$

Thus basis vectors resulting from orthogonal right-handed curvilinear-coordinates work just the same as Cartesian coordinates and are just as useful.

The additional complexity arises when we want to consider quantities such as $\boldsymbol{\nabla}\cdot\mathbf{v}$ and $\boldsymbol{\nabla}\cdot\mathbf{v}$. Firstly, we do not know the expression for the decomposition of $\boldsymbol{\nabla}$ in terms of $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$ and $\hat{\mathbf{e}}_w$, so we need to work this out. Secondly, the basis vectors $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$ and $\hat{\mathbf{e}}_w$ are now position dependent and so the partial derivatives in $\boldsymbol{\nabla}$ will act on them. We thus have to be very careful with the order in which differential operators, components and vectors appear. These complexities are best addressed within specific examples, as they differ for each choice of curvilinear coordinates.

Let us calculate the decomposition of $\boldsymbol{\nabla}$ in terms of basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ corresponding to the set of polar coordinates (r, ϕ) in 2D. These are orthogonal curvilinear coordinates and so all of the above applies. Explicitly we have $\mathbf{r} = (x, y)^T$, $\hat{\mathbf{r}} = (x, y)^T/r$ and $\hat{\boldsymbol{\phi}} = (-y, x)^T/r$. One could verify this by drawing the vectors and making geometrical arguments. Alternatively, one could calculate the derivatives $\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r}$ and $\boldsymbol{\phi} = \frac{\partial \mathbf{r}}{\partial \phi}$ and normalise to get $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

One simple approach is to deduce an expression for the gradient $\boldsymbol{\nabla} f$ of an arbitrary scalar field f by the knowledge that, for some small differential $\mathrm{d}\mathbf{l}$ induced by some small changes $\mathrm{d}r$ and $\mathrm{d}\phi$, we must have $\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla} f = \mathrm{d}f$. We know that the total differential has the form $\mathrm{d}f = \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \phi}\mathrm{d}\phi$. We also have that $\mathrm{d}\mathbf{l} = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\phi\hat{\boldsymbol{\phi}}$. To see this one could use geometrical arguments. Alternatively, one could find $\mathrm{d}\mathbf{l} = \mathbf{J}(\mathrm{d}r, \mathrm{d}\phi)^T$, where $\mathbf{J}$ is the Jacobian matrix that relate changes in one set of variables to another, and then decompose this in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$. The only unknown in $\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla} f = \mathrm{d}f$ is thus $\boldsymbol{\nabla} f$. We write $\boldsymbol{\nabla} f = a\hat{\mathbf{r}} + b\hat{\boldsymbol{\phi}}$ for some a and b, insert everything into $\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla} f = \mathrm{d}f$, and note that for this to be true for all changes $\mathrm{d}r$ and $\mathrm{d}\phi$, terms involving $\mathrm{d}r$ and $\mathrm{d}\phi$ must be separately equal (try it). The result is $a = \frac{\partial f}{\partial r}$ and $b = \frac{1}{r}\frac{\partial f}{\partial \phi}$, or together

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Furthermore, we can consistently define a decomposition of the differential vector operator grad in this basis as

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi}.$$

Note the ordering of terms that we must use because of the position dependence of the basis vectors. Similar reasoning to the above leads us to (try it)

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z},$$
$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

for cylindrical and spherical coordinates in 3D.

We already know what $\boldsymbol{\nabla} f$ looks like in our curvilinear basis as we used it in our derivation of the decomposition of $\boldsymbol{\nabla}$ in terms of the basis vectors. The next thing we want to do is to use our expression for $\boldsymbol{\nabla}$ to obtain expressions for $\boldsymbol{\nabla} \cdot \mathbf{v}$, $\boldsymbol{\nabla} \times \mathbf{v}$ and $\nabla^2 f$ in terms of curvilinear basis vectors. This will involve acting differential operators on basis vectors, so we need to know how they act. We return to 2D polar coordinates as our specific example. There we had $\hat{\mathbf{r}} = (x, y)^T/r$ and $\hat{\boldsymbol{\phi}} = (-y, x)^T/r$, or equivalently $\hat{\mathbf{r}} = (\cos\phi, \sin\phi)^T$ and $\hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi)^T$, from which we can work out all of the possible ways a differential operator could act on a basis vector: $\frac{\partial \hat{\mathbf{r}}}{\partial r} = \frac{\partial \hat{\boldsymbol{\phi}}}{\partial r} = 0$, $\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = \hat{\boldsymbol{\phi}}$, $\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\mathbf{r}}$. One can also derive these geometrically. We are now in a position to work out the expressions we are after e.g.

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi}\right) \cdot \left(v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}}\right) = \hat{\mathbf{r}} \cdot \left(\frac{\partial v_r}{\partial r}\hat{\mathbf{r}} + \frac{\partial v_\phi}{\partial r}\hat{\boldsymbol{\phi}}\right) + \frac{1}{r}\hat{\boldsymbol{\phi}} \cdot \left(\frac{\partial v_r}{\partial \phi}\hat{\mathbf{r}} + v_r\hat{\boldsymbol{\phi}} + \frac{\partial v_\phi}{\partial \phi}\hat{\boldsymbol{\phi}} - v_\phi\hat{\mathbf{r}}\right),$$
$$= \frac{\partial v_r}{\partial r} + \frac{1}{r}v_r + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} = \frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi}.$$

The same procedure can be used to work out how differential operators act on basis vectors for cylindrical and spherical coordinates in 3D, and to obtain expressions for $\boldsymbol{\nabla} \times \mathbf{v}$ and $\nabla^2 f$ also.

8. Work out $\boldsymbol{\nabla} \times \mathbf{v}$ in terms of basis vectors corresponding to cylindrical polar coordinates (ρ, ϕ, z).

There are many derivatives and integrals that are more naturally calculated using curvilinear coordinates. Try a few of them. It is often equally possible to calculate them using Cartesian coordinates. It is also possible to reproduce several of the calculations above using curvilinear coordinates. A curious student might be interested to try this and see the difference in ease.

9. The magnetic field $\mathbf{B}(\mathbf{r})$ at a distance r from a straight wire carrying a current I has magnitude $\mu_0 I/2\pi r$. The lines of force are circles centred on the wire and in planes perpendicular to it. Show that $\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$.

10. $*$ Evaluate $\iint_{\mathcal{S}} \mathbf{r} \cdot \mathrm{d}\mathbf{S}$ over the surface of a sphere of radius a centred at the origin.

11. $*$ Calculate the solid angle of a cone of half-angle α. The solid angle of a surface $\mathcal{S}$ as seen from the origin can be defined as $\Omega = \iint_{\mathcal{S}} \mathrm{d}\Omega = \iint_{\mathcal{S}} \mathbf{r} \cdot \mathrm{d}\mathbf{S}/r^3$. To answer this question you will have to choose an appropriate surface.

12. $*$ If you have time, try as many of the class problems from this and the previous tutorial on vector calculus as you can. Practice makes perfect.

Class problems

13. A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\mathbf{v}(\mathbf{r})$ as if it were a rigid body. Calculate $\nabla \cdot \mathbf{v}$ and interpret the result. Calculate $\nabla \times \mathbf{v}$. Can the flow be represented in terms of a velocity potential ϕ such that $\mathbf{v} = \nabla\phi$? If so, what is ϕ?

14. Evaluate $\int \mathbf{r} \cdot \mathrm{d}\mathbf{S}$ over the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$.

15. Show that the surface area of the curved portion of a hemisphere of radius a is $2\pi a^2$ by

(i) directly integrating the element of area $a^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$ over the surface of the hemisphere,

(ii) and projecting onto an integral taken over the x-y plane.

16. Use $\boldsymbol{\nabla} \times (\phi\mathbf{F}) = \phi(\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \times (\nabla\phi)$ and Stokes' theorem to prove in the case when $\mathbf{F} = \nabla\psi$, where ψ is another scalar field, that

$$\oint_{\mathcal{C}} (\phi\mathbf{F}) \cdot \mathrm{d}\mathbf{l} = \int_{\mathcal{S}} ((\nabla\phi) \times (\nabla\psi)) \cdot \mathrm{d}\mathbf{S}\,.$$

Verify this for the case when $\phi = x^2 y$, $\psi = xz$ and the line integral is along the curve $z = 1$, $x^2 + y^2 = 1$.

17. Use Stoke's theorem to show $\iint_{\mathcal{S}} \mathrm{d}\mathbf{S} \times \boldsymbol{\nabla}\phi = \oint_{\mathcal{C}} \phi\mathrm{d}\mathbf{l}$ and $\iint_{\mathcal{S}} (\mathrm{d}\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{v} = \oint_{\mathcal{C}} \mathrm{d}\mathbf{l} \times \mathbf{v}$, where $\mathcal{C}$ is the boundary to $\mathcal{S}$.

18. Use the divergence theorem to show $\iiint_{\mathcal{V}} \boldsymbol{\nabla}\phi \mathrm{d}V = \oiint_{\mathcal{S}} \phi\mathrm{d}\mathbf{S}$ and $\iiint_{\mathcal{V}} (\boldsymbol{\nabla} \times \mathbf{v})\mathrm{d}V = \oiint_{\mathcal{S}} \mathrm{d}\mathbf{S} \times \mathbf{v}$, where $\mathcal{S}$ is the boundary to $\mathcal{V}$.

19. Show that (a) a vector surface satisfies $\mathbf{S} = \iint_{\mathcal{S}} \mathrm{d}\mathbf{S} = \frac{1}{2}\oint_{\mathcal{C}} \mathbf{r} \times \mathrm{d}\mathbf{l}$, where $\mathcal{C}$ is the boundary to $\mathcal{S}$, and (b) a volume satisfies $V = \iint_{\mathcal{V}} \mathrm{d}V = \frac{1}{3}\oiint_{\mathcal{S}} \mathbf{r} \cdot \mathrm{d}\mathbf{S}$, where $\mathcal{S}$ is the boundary to $\mathcal{V}$.