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Keble College - Hilary 2015

CP3&4: Mathematical methods I&II

Tutorial 3 - Waves and normal modes

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 3rd week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: DJM, FR 3, 7 and 8. Additional problem sets: NWI, NWII and NWIII.

Goals

- Understand ideal waves and the ideal wave equation.
- Learn how to describe their reflection and transmission at boundaries.
- Understand dispersion, group and phase velocity, and how wave packets evolve.
- Start to get an idea of the importance and usefulness of Fourier series and transformations.
- Understand the role that boundary conditions play in restricting the solutions to the wave equation.

Problems

In previous years, students have found the topic of waves difficult. Not because they don't know a wave when they see it, or because they find specific questions difficult. Rather, it is because the topic of waves is so broad and diverse that it is hard to know one's place in the topic, and order, draw similarities between or contrast the various different physical systems and effects exhibited by them. It is for this reason that I have prepared some notes to accompany the two waves problem sets that try and introduce the mathematical concept of waves in stages of increasing complexity, away from the physical systems. This should complement what is normally found in lecture notes and add to what is found here.

The story starts with an ideal wave, which is a solution to the ideal wave equation. Various systems are, under some approximations, governed by the ideal wave equation and exhibit ideal waves. The go-to example for this course is 1D and will serve as our intuition-builder and visualiser. It is a string of density ρ and tension T, with transverse displacement $y(x,t)$ at position x and time t. Let's start by considering this system.

1. Let $y(x,t)$ be the displacement of a string of density ρ and tension T at position x and time t.

(a) Show that, under suitable approximations, the displacement obeys the wave equation $c^2\partial_x^2 y = \partial_t^2 y$, giving an expression for c.

(b) What is the general solution (d'Alembert's solution) to this wave equation? What does c represent?

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. G.G. Ross and past Oxford Prelims exam questions.

(c) Show that the kinetic energy U and the potential energy V of a length $\lambda = 2\pi/k$ are given by

$$U = \int_0^\lambda \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x, \qquad \text{and} \qquad V = \int_0^\lambda \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\,.$$

Evaluate these for the ideal sinusoidal wave $y = A\cos(kx+\omega t+\phi)$ and show that $U = V$ over a whole wavelength.

Knowing the general solution, it is possible to obtain a specific solution for each set of initial conditions. We are familiar with this for ODEs, but need to get used to this for PDEs also. Understand this through two examples.

2. At time $t = 0$, a string is initially at rest with

$$y(x,0) = \sin(\pi x/a) \quad \text{for} \quad -a \le x \le a\,, \quad \text{and} \quad y(x,0) = 0 \quad \text{otherwise}\,.$$

Using d'Alembert's solution to the wave equation $c^2\partial_x^2 y = \partial_t^2 y$, find and sketch the displacement $y(x,t)$ of the string at $t = 0$, $t = a/2c$, and $t = a/c$.

3. * Two transverse, initially non-overlapping, wavepackets move on the same piece of string. The first has displacement y non-zero only for $kx+\omega t$ between π and 2π, when it is equal to $y = A\sin(kx+\omega t)$. The second has $y = A\sin(kx-\omega t)$ for $kx-\omega t$ between -2π and $-\pi$, and is zero otherwise. At $t = 0$, the displacement is as shown in figure 1. What is the displacement of the string at $t = 3\pi/2\omega$? Calculate and compare the energy of the string at time $t = 0$ and $t = 3\pi/2\omega$. Is this what you expect?

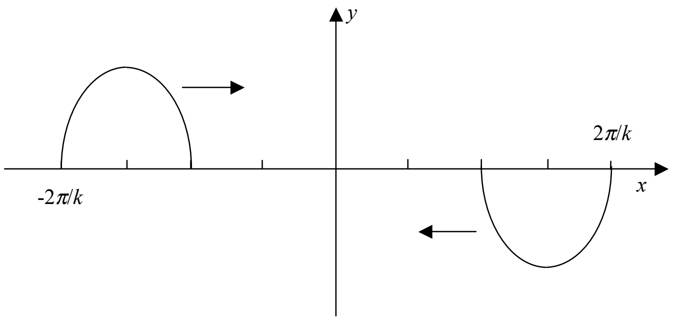

Figure 1: Two travelling wavepackets.

We have so far considered an infinite homogeneous string and its solutions. We can also considered inhomogeneities. The first way is to consider two semi-infinite regions of string with different ρ and T joined at a boundary. It is possible to solve such a system by finding the general solution in each region, and imposing reasonable boundary conditions at the boundary.

4. Two semi-infinite strings are connected at $x = 0$ and stretched to a tension T. They have linear densities ρ_1 and ρ_2 respectively. A harmonic travelling wave, represented by the complex function

$$y(x,t) = A\mathrm{e}^{\mathrm{i}\omega(x/v_1 - t)}\,,$$

travels along string 1 towards the boundary at $x = 0$. Write the general solution. State and justify the boundary conditions. Determine the amplitudes of the reflected and transmitted waves. Comment on reflection and transmission in the different limits of ρ_1/ρ_2, interpreting this physically. Does the shape of the incoming wavepacket matter?

Check that these amplitudes are such that energy conservation in the region at $x \approx 0$ is obeyed.

5. $*$ An infinite string lies along the x-axis, and is under tension T. It consists of a section at $0 < x < a$ of linear density ρ_1 and two semi-infinite pieces of density ρ_2 on either side. A moving wave $A\cos(2\pi(x/\lambda + \nu t))$ travels along the string at $x > a$ towards the short section. Here ν is the frequency and λ the wavelength of the wave.

How many types of waves are there in the various sections of the string? How many boundary conditions need to be satisfied?

Show that, if $a = n\lambda_1$ (where λ_1 is the wavelength on the short section, and n is an integer), the amplitude of the wave that emerges at $x < 0$ is A. What is the amplitude of the wave in the short section?

A further way of introducing inhomogeneity is to attach a mass at a boundary, i.e. create an impedance. The appropriate boundary conditions are different in this case, which alters the reflection and transmission properties.

6. An infinite string of linear density ρ is under tension T and has a mass m connected at $x = 0$. Calculate the amplitude reflection coefficient for transverse waves incident on the mass. What is the value of the amplitude reflection coefficient as m tends to zero? Does the shape of the incoming wavepacket matter?

Throughout the above we have been dealing with the ideal wave equation and ideal waves, where a sinusoidal wave with wavevector k will oscillate at frequency $\omega(k) = ck$ where c is a constant, defining a speed. More generally, in a dispersive medium, the wave equation supports sinusoidal waves with wavevectors k corresponding to frequency $\omega(k) \neq ck$. The quantities $v = \omega(k)/k$, $g = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}$ and $c = \lim_{k\to 0} \omega(k)/k$ are all different, controlling different aspects of the physics.

7. What is meant by a dispersive medium, phase velocity v and group velocity g? Roughly explain the effect dispersion has on the shape of a wavepacket.

8. $*$ Explain the expressions for phase velocity v and group velocity g by appealing to the example of two travelling waves of slightly different wavevectors k.

9. $*$ In quantum mechanics, a particle of momentum p and energy E has associated with it a wave of wavelength λ and frequency f given by

$$\lambda = \frac{h}{p} \quad \text{and} \quad f = \frac{E}{h}\,,$$

where h is Planck's constant. Find the phase and group velocities of these waves when the particle (a) is non-relativistic, given that

$$p = m_0 v \quad \text{and} \quad E = \frac{m_0 v^2}{2}\,,$$

and (b) is relativistic, in which case

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}\,.$$

The particle's rest mass is m_0, its speed is v, and the speed of light is c. Comment on your answers.

Class problems

So far we have only considered infinite or semi-infinite systems. However, real systems are finite and we must specific some boundary conditions to describe the behaviour at the boundaries to the system. For large systems, away from the boundaries, this choice won't matter and we'll arrive back at the mathematics of infinite systems. However, we should know how to tackle boundary conditions and study finite systems. Consider this in the first class problem.

10. (a) A string of uniform linear density ρ is stretched to a tension T, its ends being fixed at $x = 0$ and $x = L$. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that $c^2\partial^2 y/\partial x^2 = \partial^2 y/\partial t^2$ where $c^2 = T/\rho$. What is meant by the statement that this equation is *linear*? Verify that

$$y(x,t) = A_r \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{r\pi ct}{L}\right), \qquad \text{and} \qquad y(x,t) = B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right).$$

where r is any integer, are both solutions of this equation, obeying the boundary conditions $y(0,t) = y(L,t) = 0$. Explain why sums of such solutions are also solutions.

(b) The string is such that at $t = 0$, $\partial y/\partial t = 0$ for all x, and $y(x,0)$ has the shape shown in figure 2, i.e. the mid-point is drawn aside a small distance a. Explain why the solution after the mid-point is released has the form

$$y(x,t) = \sum_{r=0}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right).$$

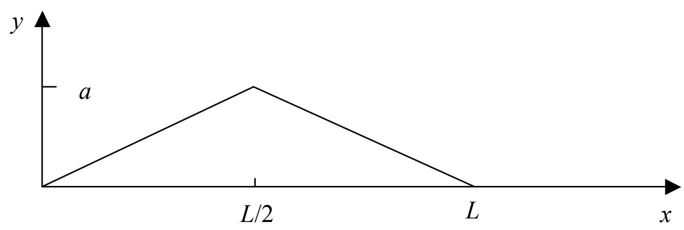

Figure 2: Initial displacement.

Remark: There are infinitely many constants $(B_1, B_2, \cdots)$ in this expression. They can be determined from the initial displacement of the string by the technique of Fourier Analysis

$$y(x,0) = \sum_r B_r \sin\left(\frac{r\pi x}{L}\right) \qquad \text{hence} \qquad B_r = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{r\pi x}{L}\right)\mathrm{d}x\,.$$

(c) Stationary waves $y(x,t) = f(x)g(t)$ exist on this string of length L. We assume that the x-dependence of the displacement is $f(x) = A\sin(kx)$. Using the general solutions above, find an expression for k. What is the t-dependence of $g(t)$? [This involves 2 arbitrary constants].

(d) At $t = 0$, the displacement is

$$y(x,0) = \sin\left(\frac{\pi x}{L}\right) + 2\sin\left(\frac{2\pi x}{L}\right),$$

and the string is instantaneously stationary. Find the displacement at subsequent times. Make rough sketches of $y(x,t)$ at the following times t: 0, $L/4c$, $L/2c$, $3L/4c$, L/c.

11. The group velocity is

$$g = \frac{\mathrm{d}\omega}{\mathrm{d}k},$$

where ω is the angular frequency and k the wave number. Show that alternative expressions for g are

$$g = v + k\frac{\mathrm{d}v}{\mathrm{d}k}, \qquad g = v - \lambda\frac{\mathrm{d}v}{\mathrm{d}\lambda}, \qquad g = \frac{c}{\mu}\left(1 + \frac{\lambda}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right),$$

where μ is the refractive index for waves of wavelength λ and wave number k in the medium and c the vacuum speed of light.

Use the above expression $g = v - \lambda\mathrm{d}v/\mathrm{d}\lambda$ to show that

$$g = v\left[1 - \frac{1}{1 + \frac{v}{\lambda'}\frac{\mathrm{d}\lambda'}{\mathrm{d}v}}\right],$$

where λ' is the wave length in vacuum.