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Keble College - Hilary 2015

CP3&4: Mathematical methods I&II

Tutorial 2 - Vector calculus and multiple integrals

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 2nd week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB 6, 10 and 11. Additional problem sets: MV.

Goals

- Understand what vector calculus is.
- Understand what the grad of scalar field, and the div and curl of a vector field are.
- Know how to evaluate integrals of more than one variable, so-called multiple integrals.
- Work out how to express physical quantities of geometric objects as multiple integrals.
- Understand exact differentials and their role in line integrals.

Problems

The fundamental objects in vector calculus are scalars ϕ and vectors $|v\rangle$ that depend on the position $|r\rangle$ in space. Such objects ϕ and $|v\rangle$ are called scalar and vector fields. The scalar field ϕ might for instance represent a potential energy of a charged particle in an electric field and $|v\rangle$ might be the electric field. Here we use a Cartesian basis $\mathcal{B} = \{|i\rangle, |j\rangle, |k\rangle\}$ in 3D, so that we may deal entirely with the column vector representations of the vectors, i.e. $\mathbf{r} = (x, y, z)^T$ and $\mathbf{v} = (v_x, v_y, v_z)^T$. Most things in electromagnetism and mechanics are in 3D, but sometimes the problem reduces to a lower dimensional one, and it's often instructive to consider such cases when we're learning, to help us see the wood through the trees. For 2D calculations we will therefore have instead $\mathcal{B} = \{|i\rangle, |j\rangle\}$, $\mathbf{r} = (x, y)^T$ and $\mathbf{v} = (v_x, v_y)^T$. We'll occasionally refer back to the trivial 1D situation, which is a bit special as a 1D vector is just a scalar, so we only have scalar fields $\phi(x)$ that depend on position x, which isn't really vector calculus. However, it'll be interesting to think about how what we are doing is an extension of the 1D case, with which we are much more familiar.

Now we have the fundamental objects, we can introduce their properties or quantities derived from them that are particularly interesting to physicists. Usually we are interested in how $\phi(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ change with respect to $\mathbf{r}$, but also sums of the values $\phi(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ take over a set of values $\mathbf{r}$. The former refers to (possibly vector) derivatives of $\phi(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$, the latter to integrals of $\phi(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ over (possibly high-dimensional) regions of space. So we will study derivatives of scalar and vector fields and their integration over regions of space, where the the region and space can both have dimension great than one. We'll start with derivatives and then consider integrals.

[1]These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. S. Rawlings and past Oxford Prelims exam questions.

The basic measure of how a scalar function $\phi(\mathbf{r})$ varies in 1D space is its derivative $\mathrm{d}\phi/\mathrm{d}x$, which is itself a scalar function. In 3D the number of options increases. We have available to us all of the partial derivatives of $\phi(\mathbf{r})$ and components $v_x(\mathbf{r})$, $v_y(\mathbf{r})$ and $v_z(\mathbf{r})$ of $\mathbf{v}(\mathbf{r})$ with respect to x, y and z, which must contain in them all the information about how the scalar and vector fields vary in space. For physical applications we are usually more interested in some combinations of these partial derivatives than others. Three combinations deserve special attention for the important role they play in physics. [We also note their 2D equivalents in square brackets.]

For a scalar field $\phi(\mathbf{r})$ there is one particularly useful combination that is most closely related to what we intuitively think of as the gradient of $\phi(\mathbf{r})$. This is the gradient (grad) vector, in Cartesian coordinates

$$\boldsymbol{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)^T, \qquad \left[\boldsymbol{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)^T, \qquad \text{2D}\right]$$

For a vector field $\mathbf{v}(\mathbf{r})$ there are two quantities of importance. The divergence (div)

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}, \qquad \left[\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}, \qquad \text{2D}\right]$$

which is a scalar, and the curl

$$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}}, \qquad \left[(\boldsymbol{\nabla}\times\mathbf{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}, \qquad \text{2D}\right]$$

which is a vector. [In 2D we either take the z-component of the above quantity to be the curl or, for the purposes of curl, extend our vector $\mathbf{v}(\mathbf{r})$ to 3D with $v_z = 0$ and $\frac{\partial}{\partial z} = 0$. The curl is then always a vector along z.]

To aid in our algebra, we usually introduce a differential vector operator, called grad

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T, \qquad \left[\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T, \qquad \text{2D}\right]$$

such that its action on a scalar, or in a scalar or vector product with a vector, is consistent with the above definitions of grad, div and curl. One must note that there are different expressions for $\phi\boldsymbol{\nabla}$, $\mathbf{v}\cdot\boldsymbol{\nabla}$ and $\mathbf{v}\times\boldsymbol{\nabla}$, because a differential operator cares about the order in which quantities appear, even in a scalar product. Commutativity properties of scalar and vector multiplication change when differential operators are present, but distributivity and associativity do not. You would do well to familiarise yourself with these new algebraic properties over the next few questions and by attempting the class problems.

The div, and column vectors representations of the grad and curl, have been expressed in terms of derivatives with respect to Cartesian coordinates. Moreover, we have focused on the components of vectors related to their decomposition in terms of associated Cartesian basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. Importantly, though it may not be obvious, the div and fundamental vectors (not basis dependent column vectors) relating to grad and curl are all independent of the choice of coordinates and can be defined without reference to any coordinates, or with reference to another set of coordinates (something we will do in the next problem set). Coordinate-independence means that the new scalar and vector fields provided by the div of a vector field, grad of a scalar field and curl of a vector field, are physically well-defined and unambiguous. It also makes it easy to understand what these quantities are by drawings and visualisations, as well as mathematics.

1. Explore the meanings of, and visualise, (i) $\boldsymbol{\nabla}\phi$, (ii) $\boldsymbol{\nabla}\cdot\mathbf{v}$, and (iii) $\boldsymbol{\nabla}\times\mathbf{v}$.

   Solution: (i) In 2D you may wish to interpret $\phi(\mathbf{r})$ as the height of a surface at position $\mathbf{r}$ and thus relate $\boldsymbol{\nabla}\phi$ to the direction and magnitude of the maximum gradient and $\mathbf{r}$. Relatedly, still in 2D, you may wish to consider contour lines of constant $\phi(\mathbf{r}) = c$ and show $\boldsymbol{\nabla}\phi$ is perpendicular to these lines. Similarly, in 3D, we have surfaces of constant $\phi(\mathbf{r}) = c$ with $\boldsymbol{\nabla}\phi$ normal to them. For (ii) and (iii) you may wish to interpret $\mathbf{v}(\mathbf{r})$ as a flow of a fluid, or a force, and draw the cases where the div and curl are zero or non-zero. We'll see more ways to interpret these quantities in the next problem set, so don't go too far down the rabbit hole, just enough to be able to visualise the quantities that you are going to be writing down in this problem set.

2. For $\mathbf{F} = (3xyz^2, 2xy^3, -x^2yz)^T$ and $\phi = 3x^2 - yz$, find

   (i) $\boldsymbol{\nabla}\phi$.
   (i) $\boldsymbol{\nabla}\cdot\mathbf{F}$,
   (i) $\boldsymbol{\nabla}\times\mathbf{F}$,
   (ii) $\mathbf{F}\cdot\boldsymbol{\nabla}\phi$,
   (iii) $\boldsymbol{\nabla}\cdot(\phi\mathbf{F})$,
   (iv) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)$.

3. $*$ Use the summation convention to prove that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0$ and $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$.

4. $*$ Find an expression using Cartesian coordinates for the Laplacian differential operator $\Delta = \nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$.

The next thing we want to do is discuss how to sum the values that scalar $\phi(\mathbf{r})$ and vector $\mathbf{v}(\mathbf{r})$ fields take over a region $\mathcal{R}$ of space. If this region $\mathcal{R}$ has a dimension higher than one, then it will take more than one variable to parameterise, meaning we will need integrals of more than one variable, called multiple integrals. We'll first get comfortable with multiple integrals and then come back to showing how they arise in vector calculus.

5. For the following integrals, sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}}\int_{v^2}^{2} v\,\mathrm{d}u\,\mathrm{d}v\,, \qquad \int_0^4\int_0^{\sqrt{u}} v\sqrt{u}\,\mathrm{d}v\,\mathrm{d}u\,, \qquad \int_0^1\int_{-v}^{v^2} u\,\mathrm{d}u\,\mathrm{d}v\,.$$

   Solution: 1, 32/5, $-1/15$.

6. $*$ Reverse the order of integration and hence evaluate

$$\int_0^{\pi}\int_v^{\pi} u^{-1}\sin(u)\,\mathrm{d}u\,\mathrm{d}v\,.$$

   Solution: 2.

We all appreciate the usefulness of changing the variable for a single integral. The same thing will be useful in multiple integrals. Let's review the single integral case so we can work out how to extend the procedure to multiple integrals. Consider some integral $I = \int_{\mathcal{R}} f(x)\,\mathrm{d}x$ *that we want to equate to an integral in some related variable* $u(x)$*. The first obvious things to do is to define a region* $\tilde{\mathcal{R}}$ *in* u *that corresponds to the region* $\mathcal{R}$ *in* x*. The second obvious thing to do is to consider the function* $f(u) = f(x(u))$*, the value taken by* $f(x)$ *at values of* x *corresponding to values of* u *(note here the sloppy labelling of two different functions by the same symbol). The third, and hardest part to understand, is that a small change* $\mathrm{d}u$ *in* u*-space will correspond to a small length* $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ *in* x*-space. So if we want to add up small changes in length corresponding to each* $\mathrm{d}u$ *multiplied by the value of the function at that the location of that length* $f(u)$ *we need to sum up terms of the form* $f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$*. Hence we have* $I = \int_{\tilde{\mathcal{R}}} f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$*. To draw analogy with what we are about to do, we might call* $\frac{\mathrm{d}x}{\mathrm{d}u}$ *the Jacobian of the transformation. Trivially, you'll see the point of this soon, we could think of this as the determinant of a trival Jacobian matrix* $J = \frac{\mathrm{d}x}{\mathrm{d}u}$*. As a mnemonic, we might write* $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ *to suggest that the latter should replace the former in the integral when changing variables. Finally, the consistency of what have done should be evident by changing variable from* x *to* u *and back again, arriving back at the starting point, since* $\frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 1$*.*

The same logic can be applied to a double integral $I = \iint_{\mathcal{S}} f(x, y)\,\mathrm{d}x\mathrm{d}y$*. We wish to change variables from* (x, y) *to* (u, v)*, where each of* x *and* y *is a function of* u *and* v*, and vice versa. We replace the region* $\mathcal{S}$ *with* $\tilde{\mathcal{S}}$ *as per usual, and write* $f(u, v) = f(x(u, v), y(u, v))$ *for the values taken by the function* $f(x, y)$ *at values of* (x, y) *corresponding to values of* (u, v)*. We now wish to quantify the area in* (x, y)*-space swept out by small changes* $(\mathrm{d}u, \mathrm{d}v)$ *in* (u, v)*-space. We know that for small enough changes this area will be a parallelogram as lines of constant* u *and* v *will appear straight and parallel on the smallest length scales. We wish to find the two vectors making up the edges of this parallelogram, so that its area can be calculated by the magnitude of their vector product. The vectors along the line of constant* v *and* u *will be* $\frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u$ *and* $\frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v$*, respectively, where we have written* $\mathbf{r} = (x, y)^T$*. The area of the parallelogram is thus* $|\frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u \times \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v| = |\mathbf{J}|\mathrm{d}u\mathrm{d}v$*, where* $|\mathbf{J}|$ *is the so-called Jacobian and equal to the determinant of the Jacobian matrix*

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

The correct transformed equation for the integral is thus $I = \iint_{\tilde{\mathcal{S}}} f(u, v)|\mathbf{J}|\,\mathrm{d}u\,\mathrm{d}v$*. Often we write* $|\mathbf{J}|$ *as* $\frac{\partial(x,y)}{\partial(u,v)}$*. We might write* $\mathrm{d}x\,\mathrm{d}y = \frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\,\mathrm{d}v$ *as a mnemonic. Again, as a consistency check, one can show that* $\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = 1$*.*

The same logic can be applied to triple integrals $\iiint_{\mathcal{V}} f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$*, where we change to variables* (u, v, w)*. Elements along lines where only one of variable* (u, v, w) *change are given by* $\frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u$*,* $\frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v$ *and* $\frac{\partial \mathbf{r}}{\partial w}\mathrm{d}w$*. Thus changes* $(\mathrm{d}u, \mathrm{d}v, \mathrm{d}w)$ *in* (u, v, w)*-space sweep out a parallelogram of volume* $\{\frac{\partial \mathbf{r}}{\partial u}\frac{\partial \mathbf{r}}{\partial v}\frac{\partial \mathbf{r}}{\partial w}\}\mathrm{d}u\mathrm{d}v\mathrm{d}w = |\mathbf{J}|\mathrm{d}u\mathrm{d}v\mathrm{d}w$ *in* (x, y, z)*-space. This gives us* $I = \iint_{\tilde{\mathcal{V}}} f(u, v, w)\frac{\partial(x,y,z)}{\partial(u,v,w)}\,\mathrm{d}u\mathrm{d}v\mathrm{d}w$ *and is summarised by the mnemonic* $\mathrm{d}x\mathrm{d}y\mathrm{d}z = |\mathbf{J}|\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{\partial(x,y,z)}{\partial(u,v,w)}\mathrm{d}u\mathrm{d}v\mathrm{d}w$*, where the Jacobian matrix is*

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

The same consistency check can be made $\frac{\partial(x,y,z)}{\partial(u,v,w)}\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$*.*

Try a few examples of this next.

7. Calculate the Jacobians $\frac{\partial(x,y)}{\partial(r,\phi)}$, $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}$ and $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ associated with change of variables from Cartesian coordinates to polar, cylindrical and spherical coordinates, respectively. You can do this by calculating the Jacobian matrix and taking the determinant, or graphically. You should try and understand both methods and their connection.

8. $*$ Using the change of variable $x+y=u$, $x-y=v$ evaluate the double integral

$$\iint_{\mathcal{S}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $\mathcal{S}$ is the region bounded by the straight lines $y=x$, $y=x+2$, $y=-x$ and $y=-x+2$.

We now go back to vector calculus and the integrals that appear there. The remaining task of this problem set is to discuss the types of integrals that appear in vector calculus, how they can be re-expressed as multiple integrals, in general and for specific examples. This a big undertaking. Let's begin by listing the types of integrals that appear in vector calculus, i.e. all the ways we can sum the values of scalar and vector fields over a region $\mathcal{R}$ of space.

In 1D there is one way in which this can occur. We must sum our scalar field $\phi(x)$ over some region $\mathcal{R}$ of the parameter x. We arrive at the simple integral $\int_{\mathcal{R}} \phi(x)\,\mathrm{d}x$, which is the limit of a sum over the value of the scalar field in that region $\phi(x)$ multiplied by the size of small elements of the region $\mathcal{R}$.

In 2D and 3D, the number of options increases significantly. Firstly, the function being summed over could be a vector not just a scalar. Secondly, in 2D we are able to sum over both one and two-dimensional regions of $\mathbf{r}$, i.e. a curve within 2D space or an area of 2D space itself. Similarly in 3D we are able to sum over both one, two and three-dimensional regions of $\mathbf{r}$, i.e. a curve or surface within 3D space or a volume of 3D space itself. Thus instead of small additions to the region of integration being just small lengths $\mathrm{d}l$ (l rather than x gets used when the line might not be along the x-axis), we will consider small areas $\mathrm{d}S$, and volumes $\mathrm{d}V$. Thirdly, in 2D and 3D, elements of a line will not always be parallel and their direction might be important. Thus we might make use of the vector lengths $\mathrm{d}\mathbf{l}$ rather than merely their magnitude. Similarly not all surfaces in 3D will be parallel and we might make use of the vector areas $\mathrm{d}\mathbf{S}$, which has magnitude $\mathrm{d}S$ and is normal to the surface.

This leaves us with several possible types of integral. In 1D we have only $\int_{\mathcal{C}} \phi(x)\,\mathrm{d}x$. In 2D we have the following possibilities

$$\int_{\mathcal{C}} \phi(\mathbf{r})\,\mathrm{d}l,\ \int_{\mathcal{C}} \mathbf{v}(\mathbf{r})\,\mathrm{d}l,\ \int_{\mathcal{C}} \phi(\mathbf{r})\,\mathrm{d}\mathbf{l},\ \int_{\mathcal{C}} \mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{l},\ \int_{\mathcal{C}} \mathbf{v}(\mathbf{r})\times\mathrm{d}\mathbf{l},\ \iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}S,\ \iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\,\mathrm{d}S,$$

where $\mathcal{C}$ is a curve in space and $\mathcal{S}$ is region of 2D space. In 3D we have the following additional possibilities

$$\iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}\mathbf{S},\ \iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{S},\ \iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\times\mathrm{d}\mathbf{S},\ \iiint_{\mathcal{V}} \phi(\mathbf{r})\,\mathrm{d}V,\ \iiint_{\mathcal{V}} \mathbf{v}(\mathbf{r})\,\mathrm{d}V,$$

where $\mathcal{S}$ is surface in space and $\mathcal{V}$ is a region of 3D space. Often, only a single rather than double or triple integral sign is used for a surface or volume integral. You should be able to understand what these integrals mean from first principles, just as you do for a normal integral. This understanding and the ability to visualise what you are calculating will be the key to successful evaluation of integrals.

The general strategy for evaluating such integrals is to parameterise the values taken by $\mathbf{r}$ *in the region* $\mathcal{R}$*. For regions that are lines* $\mathcal{C}$*, surfaces* $\mathcal{S}$ *and volumes* $\mathcal{V}$ *we need one, two and three parameters, respectively. We denote such parameters* u*,* (u, v) *and* (u, v, w) *and thus have* $\mathbf{r}(u)$*,* $\mathbf{r}(u, v)$ *and* $\mathbf{r}(u, v, w)$*, respectively. If you feel uncomfortable starting so generally, you might want to chose* u*,* v*, and* w *from* x*,* y*, and* z *during your first reading, before trying to generalise to other familiar coordinates e.g.* (ρ, ϕ, z)*.*

We first use the fact that $\mathbf{r}(u, v, w)$ *to write the values of the scalar* $f(\mathbf{r})$ *or vector function* $\mathbf{v}(\mathbf{r})$ *at points* $\mathbf{r}$ *in the region* $\mathcal{R}$ *as functions* $f(u, v, w) = f(\mathbf{r}(u, v, w))$ *and* $\mathbf{v}(u, v, w) = \mathbf{v}(\mathbf{r}(u, v, w))$ *of the parameters* (u, v, w) *(mutatis mutandis for lines and surfaces, when only one or two parameters are needed). Notice again the sloppy use of the same symbol for different functions.*

We second calculate the differentials, line $\mathrm{d}\mathbf{l}$*, surface* $\mathrm{d}\mathbf{S}$ *and volume* $\mathrm{d}V$ *elements swept over by small changes* $(\mathrm{d}u, \mathrm{d}v, \mathrm{d}w)$ *in the parameters* (u, v, w)*. Specifically, for line elements we have simply that the line swept over by a small change* $\mathrm{d}u$ *is* $\mathrm{d}\mathbf{l} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u$*.*

For surface elements the area swept over will be a parallelogram. To see this, think about lines in which only one of the parameters u *or* v *can change. On a small enough length scales such lines will be straight and parallel for each of* u *or* v*. The surface will we want will be a parallelogram formed by these lines. Two vectors describing the sides of the parallelogram will be* $\frac{\partial\mathbf{r}}{\partial u}\mathrm{d}u$ *and* $\frac{\partial\mathbf{r}}{\partial v}\mathrm{d}v$*. The vector that is normal to the surface element with magnitude equal to its volume is thus* $\mathrm{d}\mathbf{S} = \frac{\partial\mathbf{r}}{\partial u}\mathrm{d}u \times \frac{\partial\mathbf{r}}{\partial v}\mathrm{d}v = \mathbf{j}\mathrm{d}u\mathrm{d}v$*, where* $\mathbf{j} = \frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}$*.*

Similar reasoning implies that volume elements are parallelepipeds and the vectors describing the edges are $\frac{\partial\mathbf{r}}{\partial u}\mathrm{d}u$*,* $\frac{\partial\mathbf{r}}{\partial v}\mathrm{d}v$ *and* $\frac{\partial\mathbf{r}}{\partial w}\mathrm{d}w$*, each running along a line where only one parameter is allowed to vary. The volume of such a parallelogram is* $\mathrm{d}V = \{\frac{\partial\mathbf{r}}{\partial u}\mathrm{d}u\frac{\partial\mathbf{r}}{\partial v}\mathrm{d}v\frac{\partial\mathbf{r}}{\partial w}\mathrm{d}w\} = |\mathbf{J}|\mathrm{d}u\mathrm{d}v\mathrm{d}w$*, where* $|\mathbf{J}| = \{\frac{\partial\mathbf{r}}{\partial u}\frac{\partial\mathbf{r}}{\partial v}\frac{\partial\mathbf{r}}{\partial w}\}$*, as appeared earlier for our Jacobian in 3D.*

In summary we have $\mathrm{d}\mathbf{l} = \mathbf{j}\mathrm{d}u$*,* $\mathrm{d}\mathbf{S} = \mathbf{j}\mathrm{d}u\mathrm{d}v$ *and* $\mathrm{d}V = |\mathbf{J}|\mathrm{d}u\mathrm{d}v\mathrm{d}w$*, where* $\mathbf{j} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}$*,* $\mathbf{j} = \frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}$ *and* $|\mathbf{J}| = \{\frac{\partial\mathbf{r}}{\partial u}\frac{\partial\mathbf{r}}{\partial v}\frac{\partial\mathbf{r}}{\partial w}\}$*. For the magnitudes of the line and surface elements we have* $\mathrm{d}l = |\mathbf{j}|\mathrm{d}u$ *and* $\mathrm{d}S = |\mathbf{j}|\mathrm{d}u\mathrm{d}v$*. The vector* $\mathbf{j}$ *and its magnitude* $|\mathbf{j}|$ *are themselves functions of* (u, v, w)*. It is worth mentioning that for a length in 1D,* $|\mathbf{j}| = |\mathbf{J}|$ *is merely the Jacobian. Similarly for an area in 2D,* $|\mathbf{j}| = |\mathbf{J}|$*. We have already mentioned that the Jacobian* $|\mathbf{J}|$ *appears for a volume in 3D. However, we do not always have* $|\mathbf{j}| = |\mathbf{J}|$ *because the region of integration could be of a lower dimension than space e.g. a line in 2D or 3D, or a surface in 3D.*

As a final step, we write the set of values of parameters (u, v, w) *over which we need to integrate as* $\tilde{\mathcal{R}}$*. Together these three steps turn any of the vector calculus integrals above into a multiple integral of a scalar of vector function of* (u, v, w)*.*

Above, we have shown, rather generally, how all integrals appearing in vector calculus can be re-expressed as multiple integrals. We'll come back to it many many times as we slowly work through how, in practice, to do this for each of the type of integrals and each of the usual choices of parameters (u, w, v)*. When specific examples of* (u, w, v) *are used, everything will be more intuitive and less mathematical. Often it is quite clear what the line, surface or volume elements are, so the multiple integral is obtained almost automatically. For example, think back to your calculation in the first few weeks of last term of the arc length of a function, which is in fact a line integral in 2D space. You quickly worked out the magnitude of the line element and re-expressed the line integral as a normal single integral. You didn't need to know any of the above mathematics. It just shows how important it will be to visualise what you are doing.*

Let's begin our journey of descibing how to reduce different types of vector calculus integrals to multiple integrals, starting with a simple type of integral. These are integrals where the region of integration has the same dimension as the space, i.e. $\int_{\mathcal{R}} \phi(x)\,\mathrm{d}x$ *in 1D,* $\iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}S$ *in 2D and* $\iiint_{\mathcal{V}} \phi(\mathbf{r})\,\mathrm{d}V$ *in 3D. We focus on the latter two because the former is well understood.*

For these integrals we find, comparing the general expressions for line, surface and volume elements, to those for the Jacobian, that $\mathrm{d}S = \frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\mathrm{d}v$ *and* $\mathrm{d}V = \frac{\partial(x,y,z)}{\partial(u,v,w)}\mathrm{d}u\mathrm{d}v\mathrm{d}w$.

Thus for integrals over regions in 2D space, using the Jacobians for Cartesian and polar coordinates, and for integrals over 3D space, using Cartesian, cylindrical polar, or spherical coordinates, we have

$$\iint_{\mathcal{S}} f(\mathbf{r})\,\mathrm{d}S = \iint_{\tilde{\mathcal{S}}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{\tilde{\mathcal{S}}} f(r,\phi)\,r\mathrm{d}r\mathrm{d}\phi,$$
$$\iint_{\mathcal{V}} f(\mathbf{r})\,\mathrm{d}V = \iint_{\tilde{\mathcal{V}}} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\tilde{\mathcal{V}}} f(\rho,\phi,z)\,\rho\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z = \iint_{\tilde{\mathcal{V}}} f(r,\theta,\phi)\,r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

We have arrived at this in quite a formal way. But the same thing can be derived much more intuitively by geometrically calculating the area of volumes of space swept over by small changes in coordinates, for a specific set of coordinates without going through a general framework first (try it). Use the results above to express the following physical quantities as vector calculus integrals, re-express them as multiple integrals, and evaluate the integrals.

9. Calculate the area of a circle, and the volumes of a cylinder and sphere.

10. Calculate the position of the centre of mass of a hemisphere of uniform density defined by surfaces $x^2 + y^2 + z^2 = R^2$, $z > 0$. Use symmetry arguments to reduce the complexity of your calculation.

We have so far addressed perhaps the simplest type of vector calculus integrals, where the region of integration has the same dimension as the space itself. Now we consider examples where the region of integration is of a lower dimension than the space itself. We start with line integrals $\int_{\mathcal{C}} \phi(\mathbf{r})\,\mathrm{d}l$, $\int_{\mathcal{C}} \phi(\mathbf{r})\,\mathrm{d}\mathbf{l}$, $\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l}$ *and* $\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \times \mathrm{d}\mathbf{l}$ *in 2D and 3D.*

To recap, the general tactic of evaluating such integrals is to parameterise the path $\mathcal{C}$ *by some parameter* u*, so that the positions along the path are* $\mathbf{r}(u)$ *for some set* $\tilde{\mathcal{C}}$ *of values* u*. Then we we can get expressions for the changes* $\mathrm{d}\mathbf{l} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u$ *and* $\mathrm{d}l = |\mathrm{d}\mathbf{l}|$ *in terms of corresponding changes* $\mathrm{d}u$ *in* u*. We also write* $\phi(u) = \phi(\mathbf{r}(u))$ *and* $\mathbf{v}(u) = \mathbf{v}(\mathbf{r}(u))$*. We thus transform both seemingly complicated line integrals into the usual single integrals*

$$\int_{\mathcal{C}} \phi(\mathbf{r})\,\mathrm{d}l = \int_{\tilde{\mathcal{C}}} \phi(u) \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| \mathrm{d}u, \qquad \int_{\mathcal{C}} \phi(\mathbf{r})\,\mathrm{d}\mathbf{l} = \int_{\tilde{\mathcal{C}}} \phi(u)\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\,\mathrm{d}u,$$

$$\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l} = \int_{\tilde{\mathcal{C}}} \mathbf{v}(u) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u, \qquad \int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \times \mathrm{d}\mathbf{l} = \int_{\tilde{\mathcal{C}}} \mathbf{v}(u) \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\mathrm{d}u.$$

If u *is chosen as* x*,* y *or* z *then you can think of this as projecting onto the* x*,* y *or* z*-axes. For curves* $\mathcal{C}$ *that are formed of straight lines, this is usually a good choice, as the straight line of the axes will have a constant relationship with that straight line of the curve and* $\mathrm{d}\mathbf{l} = \hat{\mathbf{j}}\mathrm{d}x/\cos\theta$*, where* θ *is the angle the line (which is along* $\hat{\mathbf{j}}$ *– sorry, I have used the same notation as for the* y*-axis basis vector) makes with the* x*-axis. Sometimes the curve* $\mathcal{C}$ *will be curved, such as the circumference of a circle. In those cases it will be a much better choice to set* u *as* ϕ*. For an anticlockwise circle at radius* r *the length element will be* $\mathrm{d}\mathbf{l} = r\mathrm{d}\phi\hat{\boldsymbol{\phi}}$*, where* $\hat{\boldsymbol{\phi}} = (-y, x)^T/r$ *is a unit vector along the circumference. A good physicist, familiar with the topic, will quickly pick the right coordinate to use, and will obtain the relevant multiple integral quickly, often through geometric, diagrammatic and physical arguments rather than looking up one of the mathematical equations above.*

There is one important thing to highlight specifically relating to integrals of the type $\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l}$*. An interesting thing occurs when* $\mathbf{v}(\mathbf{r})$ *can be written as* $\mathbf{v}(\mathbf{r}) = \boldsymbol{\nabla}\phi(\mathbf{r})$ *for some* $\phi(\mathbf{r})$*. If this is the case then* $\mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l} = \boldsymbol{\nabla}\phi(\mathbf{r})\mathrm{d}\mathbf{l} = \mathrm{d}\phi$ *is simply the total differential associated with a small change* $\mathrm{d}\phi$ *in* $\phi(\mathbf{r})$*. It is said that* $\mathrm{d}\phi = \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l}$ *is an exact differential. In this case we know in advance that the integral must satisfy* $\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l} = \phi(\mathbf{r}_f) - \phi(\mathbf{r}_i)$*, where* $\mathbf{r}_i$ *and* $\mathbf{r}_f$ *are the points at which the path* $\mathcal{C}$ *starts and ends. This means that* $\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l}$ *depends only on the start and end points of the path, not on the exact path taken. Moreover, any path that is a loop, i.e. has the same start and end point, will result in the integral being zero, which is often written* $\oint_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l} = 0$*. Conversely, it is also possible to argue (think about it) that iff* $\int_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l}$ *between any two points is always independent of the path, for all paths, initial and end points, or if* $\oint_{\mathcal{C}} \mathbf{v}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l} = 0$ *for any loop* $\mathcal{C}$*, then* $\mathbf{v}(\mathbf{r}) = \boldsymbol{\nabla}\phi(\mathbf{r})$ *for some* $\phi(\mathbf{r})$*. However, showing the equivalence of the integral for a few paths and a few start and end points is only enough to suggest that this might be the case, but not prove it.*

A vector field $\mathbf{v}(\mathbf{r})$ *with the above property is called conservative for the following reason. In physics, we often have a conservative force* $\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla}V(\mathbf{r})$ *associated with a potential energy* $V(\mathbf{r})$*. We know that the energy gained by a particle moving in this force is* $\Delta E = \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{l} = V(\mathbf{r}_i) - V(\mathbf{r}_f)$*, just the difference in potential energies, an incredibly useful property that allows us to forget about the path. In the next problem set we will connect the conservative property of a field* $\mathbf{v}(\mathbf{r})$ *to it being irrotational* $\boldsymbol{\nabla} \times \mathbf{v}(\mathbf{r}) = 0$*, and connect this to surface integrals.*

11. A vector field $\mathbf{F}(\mathbf{r})$ is defined by its components (with $\mathbf{r} = (x, y, z)^{\mathrm{T}}$)

$$\mathbf{F}(\mathbf{r}) = (3x^2 + 6y, -14yz, 20xz^2)^{\mathrm{T}}.$$

Evaluate the line integral $\int \mathbf{F}\cdot\mathrm{d}\mathbf{l}$ from $(0,0,0)^{\mathrm{T}}$ to $(1,1,1)^{\mathrm{T}}$ along the following paths

(a) $x = t$, $y = t^2$, $z = t^3$;

(b) on the path of straight lines joining $(0,0,0)^{\mathrm{T}}$, $(1,0,0)^{\mathrm{T}}$, $(1,1,0)^{\mathrm{T}}$, and $(1,1,1)^{\mathrm{T}}$ in turn;

(c) the straight line joining the two points.

Is $\mathbf{F}$ conservative?

12. * With $\phi = 2xyz^2$, $\mathbf{F} = (xy, -z, x^2)$ and C the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals

$$\int_C \phi\,\mathrm{d}\mathbf{r}\,, \qquad \int_C \mathbf{F}\times\mathrm{d}\mathbf{r}\,.$$

Solution: $(8/11, 8/10, 1)$, $(-9/10, -2/3, 7/5)$.

We now move on to the next common type of integral where the region of integration has a dimension less than that of space. These are surface integrals in 3D, i.e. $\iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}S$, $\iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}\mathbf{S}$, $\iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{S}$ *and* $\iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\times\mathrm{d}\mathbf{S}$.

To recap, the general tactic of evaluating such integrals is to parameterise the surface $\mathcal{S}$ *by some parameters* (u,v)*, so that the positions along the path are* $\mathbf{r}(u,v)$ *for some set* $\tilde{\mathcal{S}}$ *of values* (u,v)*. Then we we can get expressions for the vectorised and scalar surface elements* $\mathrm{d}\mathbf{S} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}v}\mathrm{d}u\mathrm{d}v = \mathbf{j}\mathrm{d}u\mathrm{d}v$ *and* $\mathrm{d}S = |\mathrm{d}\mathbf{S}|$ *in terms of corresponding changes* $(\mathrm{d}u,\mathrm{d}v)$ *in* (u,v)*. We also write* $\phi(u,v) = \phi(\mathbf{r}(u,v))$ *and* $\mathbf{v}(u,v) = \mathbf{v}(\mathbf{r}(u,v))$*. We thus transform both seemingly complicated surface integrals into simple integrals*

$$\iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}S = \int_{\tilde{\mathcal{S}}} \phi(u,v)\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}v}\right|\mathrm{d}u\mathrm{d}v,\quad \iint_{\mathcal{S}} \phi(\mathbf{r})\,\mathrm{d}\mathbf{S} = \iint_{\tilde{\mathcal{S}}} \phi(u,v)\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}v}\,\mathrm{d}u\mathrm{d}v,$$
$$\iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\cdot\mathrm{d}\mathbf{S} = \iint_{\tilde{\mathcal{S}}} \mathbf{v}(u,v)\cdot\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}v}\right)\mathrm{d}u\mathrm{d}v,\quad \iint_{\mathcal{S}} \mathbf{v}(\mathbf{r})\times\mathrm{d}\mathbf{S} = \iint_{\tilde{\mathcal{S}}} \mathbf{v}(u,v)\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}v}\right)\mathrm{d}u\mathrm{d}v.$$

If (u,v) *is chosen as some pair of* x*,* y *or* z *then you can think of this as projecting onto one of the Cartesian planes. For surfaces* $\mathcal{S}$ *that are formed of planes, this is usually a good choice, as the slope of the Cartesian plane will have a constant relationship with that of the plane of integration and* $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\mathrm{d}x\mathrm{d}y/\cos\theta$*, where* θ *is the angle between the norm* $\hat{\mathbf{n}}$ *of the plane* $\mathcal{S}$ *and the norm of the* xy*-plane being projected on to. Sometimes the surface* $\mathcal{S}$ *will be curved, such as the surface of sphere. In those cases it will be a much better choice to set* (u,v) *as* (θ,ϕ) *in spherical coordinates. For an outward directed surface at radius* r *the surface element will be* $\mathrm{d}\mathbf{S} = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi\hat{\mathbf{r}}$*, where* $\hat{\mathbf{r}} = (x,y,z)^T/r$ *is a unit vector in the radial direction. For the curved surface of a cylinder, we would instead set* (u,v) *as* (ρ,z) *in cylindrical coordinates. For an outward directed surface at radius* r *the surface element will be* $\mathrm{d}\mathbf{S} = r\mathrm{d}\phi\mathrm{d}z\hat{\boldsymbol{\rho}}$*, where* $\hat{\boldsymbol{\rho}} = (x,y,0)^T/\rho$ *is radial along the* xy*-plane. A good physicist, familiar with the topic, will quickly pick the right coordinate to use, and will obtain the relevant multiple integral quickly, often through geometric, diagrammatic and physical arguments rather than looking up one of the mathematical equations above.*

Here we will start with a few simple examples and, next time, move on to more complicated surface integrals, using simplifying techniques, in the next tutorial.

13. Find the areas of a cylinder and sphere.

14. * Find by integration the area of the plane $x-2y+5z = 13$ cut out by the cylinder $x^2+y^2 = 9$. Use geometrical arguments to check your result.

Class problems

15. Which of the following can be described by vectors: (a) temperature, (b) magnetic field, (c) acceleration, (d) force, (e) molecular weight, (f) area, (g) angle of polarization?

16. To what scalar or vector quantities do the following expressions in suffix notation correspond (expand and sum where possible): $a_i b_j c_i$, $a_i b_j c_j d_i$, $\delta_{ij} a_i a_j$, $\delta_{ij}\delta_{ij}$, $\epsilon_{ijk} a_i b_k$, and $\epsilon_{ijk}\delta_{ij}$.

17. Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)^T$.

18. Using the summation convention, find the grad of the following scalar functions of position $\mathbf{r} = (x, y, z)$: (a) $|\mathbf{r}|^n$, (b) $\mathbf{a}\cdot\mathbf{r}$. Again using the summation convention, find the div and curl of the following vector functions (c) $\mathbf{r}$, (d) $|\mathbf{r}|^n\mathbf{r}$, (e) $(\mathbf{a}\cdot\mathbf{r})\mathbf{b}$, and (f) $\mathbf{a}\times\mathbf{r}$. Here, $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.

19. Sketch the vector fields $\mathbf{F}_1 = (x, y, 0)$ and $\mathbf{F}_2 = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

20. Consider a 3D system. If $F(\mathbf{r})$ and $G(\mathbf{r})$ are scalar fields, find and general expression for $\boldsymbol{\nabla}(F+G)$ and $\boldsymbol{\nabla}(FG)$. Evaluate this for the specific case of $F(\mathbf{r}) = x^2 z + \mathrm{e}^{y/x}$ and $G(\mathbf{r}) = 2z^2 y - xy^2$ at the point $\mathbf{r} = (1, 0, -2)^T$.

21. The pair of variables (x, y) are each functions of the pair of variables (u, v) and *vice versa*. Consider the Jacobian matrices

$$\mathbf{J}_{xy|uv} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \qquad \text{and} \qquad \mathbf{J}_{uv|xy} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

(a) Show using the chain rule that the product $\mathbf{J}_{xy|uv}\mathbf{J}_{uv|xy}$ of these two matrices equals the identity matrix $\mathbf{1}$.

(b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, φ).

(c) Recalling the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

22. Mass is distributed evenly in the positive x region of the xy-plane and in the shape of a semi-circle of radius a, centered on the origin. The mass per unit area is k. Find, using plane polar coordinates,

(i) the total mass M,

(ii) the coordinates $(\bar{x}, \bar{y})$ of the centre of mass,

(iii) the moments of inertia about the x and y axes.

Solution: $M = \pi k a^2/2$, $(\bar{x}, \bar{y}) = (4a/3\pi, 0)$, $I_x = I_y = a^4 k\pi/8$.

23. Do as in problem **22** for a semi-infinite sheet with mass per unit area

$$\sigma = k\mathrm{e}^{-(x^2+y^2)/a^2} \quad \text{for} \quad x \geq 0\,, \qquad \sigma = 0 \quad \text{for} \quad x < 0\,,$$

where a is a constant. Comment on the comparisons between these answers and those obtained in problem **22**.

Note that (see your data sheet)

$$\int_0^\infty \mathrm{e}^{-\lambda u^2}\,\mathrm{d}u = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,.$$

Solution: $M = \pi k a^2/2$, $(\bar{x}, \bar{y}) = (a/\sqrt{\pi}, 0)$, $I_x = I_y = a^4 k\pi/4$.

24. Evaluate the following integral

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\arctan(y/x)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Solution: $\pi^2 a^4/32$.

25. Calculate the position of the centre of mass of an object with a conical base and a rounded top which is bounded by the surfaces $z^2 = x^2+y^2$, $x^2+y^2+z^2 = R^2$, $z > 0$ and whose density is uniform.

26. The thermodynamic relation $\delta q = C_V \mathrm{d}T + (RT/V)\mathrm{d}V$ is not an exact differential. Show that by dividing this equation by T, it becomes exact.

27. A uniform lamina is made of the part of the plane $x+y+z=1$ lying in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

28. With $\hat{\mathbf{n}}$ the unit normal to the surface S, evaluate $\int \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ for the following cases:

(i) $\mathbf{F} = (x+y^2, -2x, 2yz)$ and S the surface of the plane $2x+y+2z=6$ in the first octant.

(ii) $\mathbf{F} = (6z, 2x+y, -x)$ and S the entire surface of the region bounded by the cylinder $x^2+z^2=9$, $x=0$, $y=0$, $z=0$ and $y=8$.

Solution: (i) 81, (ii) 18π.