Vector calculus in curvilinear coordinates

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Goals:

- Understand the difference between coordinates and vector components
- Understand the implications of basis vectors in curvilinear coordinates not being constant
- Learn how to use curvilinear coordinate systems in vector calculus

Coordinate systems and vector fields

Coordinate systems

A point in coordinate space $\mathbf{r}$ is often represented as $\mathbf{r} = (x, y, z)^T$ with x, y, and z the distances along the three coordinate axes. We can equally introduce cylindrical polar coordinates which we will use here as the prime example for curvilinear coordinate systems. They are defined through the relations

$$x = \rho\cos(\varphi), \qquad y = \rho\sin(\varphi), \quad \text{and} \quad z = z.$$

The point in space is now written as $\mathbf{r} = (\rho\cos(\varphi), \rho\sin(\varphi), z)^T$ where the meaning of the components is *still* that they give the distances along the three coordinate axes.

Vector fields

A vector field assigns a vector to each point $\mathbf{r}$ and is usually denoted as $\mathbf{F}(\mathbf{r})$ or simply $\mathbf{F}$. The vector field is often defined through components $F_i(\mathbf{r})$ which are the projections of the vector onto the three coordinate axes. For instance $\mathbf{F} = (-y, x, 0)^T/\sqrt{x^2+y^2}$ assigns vectors as indicated in figure 1a). Using cylindrical polar coordinates this vector field is given by $\mathbf{F} = (-\sin(\varphi), \cos(\varphi), 0)^T$. Here we have rewritten the vector field in different coordinates but *not* changed the meaning of its components.

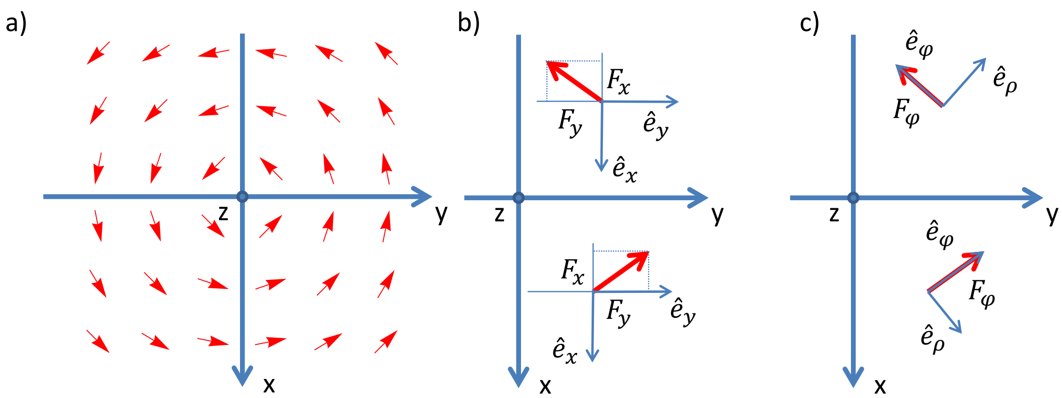

Figure 1: (a) Vector field $\mathbf{F} = (-y, x, 0)^T/\sqrt{x^2+y^2}$ (red arrows) and showing cartesian (b) and cylindrical polar (c) components for two of its vectors.

Basis vectors

Each point in coordinate space has a vector space associated with it where the vectors of vector fields live. A coordinate system $\{x_i\}$ allows us to define bases for all of these vector spaces through

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x_i}\,.$$

For cartesian coordinates the normalized basis vectors are $\hat{\mathbf{e}}_x = \hat{\mathbf{i}}$, $\hat{\mathbf{e}}_y = \hat{\mathbf{j}}$, and $\hat{\mathbf{e}}_z = \hat{\mathbf{k}}$ pointing along the three coordinate axes. They are orthogonal, normalized and *constant*, i.e. their direction does not change with the point $\mathbf{r}$[1].

Next we calculate basis vectors for a curvilinear coordinate systems using again cylindrical polar coordinates. They are given by

$$\hat{\mathbf{e}}_\rho = \frac{\partial \mathbf{r}}{\partial \rho} = \cos(\varphi)\hat{\mathbf{e}}_x + \sin(\varphi)\hat{\mathbf{e}}_y \,, \qquad \hat{\mathbf{e}}_\varphi = \frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \varphi} = -\sin(\varphi)\hat{\mathbf{e}}_x + \cos(\varphi)\hat{\mathbf{e}}_y \,, \quad \text{and} \quad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \,.$$

The $1/\rho$ in the definition of $\hat{\mathbf{e}}_\varphi$ is required for the vector to be properly normalized to 1. These basis vectors are mutually orthogonal and normalized. However, they are *not* constant, their direction changes with position. By inverting this set of linear equations for the basis vectors we find

$$\hat{\mathbf{e}}_x = \cos(\varphi)\hat{\mathbf{e}}_\rho - \sin(\varphi)\hat{\mathbf{e}}_\varphi \,, \qquad \hat{\mathbf{e}}_y = \sin(\varphi)\hat{\mathbf{e}}_\rho + \cos(\varphi)\hat{\mathbf{e}}_\varphi \,, \quad \text{and} \quad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \,.$$

We can now write a vector field $\mathbf{F}(\mathbf{r}) = (F_x(\mathbf{r}), F_y(\mathbf{r}), F_z(\mathbf{r}))_c^T \equiv F_x(\mathbf{r})\hat{\mathbf{e}}_x + F_y(\mathbf{r})\hat{\mathbf{e}}_y + F_z(\mathbf{r})\hat{\mathbf{e}}_z$ in new components using the above relations between the basis vectors for different coordinate systems. This is still the *same* vector field but now written as $\mathbf{F}(\mathbf{r}) = (F_\rho(\mathbf{r}), F_\varphi(\mathbf{r}), F_z(\mathbf{r}))_p^T \equiv F_\rho(\mathbf{r})\hat{\mathbf{e}}_\rho + F_\varphi(\mathbf{r})\hat{\mathbf{e}}_\varphi + F_z(\mathbf{r})\hat{\mathbf{e}}_z$. The new components are projections on the basis vectors $\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\varphi$, and $\hat{\mathbf{e}}_z$. We use indices c and p to make explicit which components are used. In detail, the change of components is carried out by

$$\begin{aligned}\mathbf{F} \quad &= \quad \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_c = F_x\hat{\mathbf{e}}_x + F_y\hat{\mathbf{e}}_y + F_z\hat{\mathbf{e}}_z = F_x[\cos(\varphi)\hat{\mathbf{e}}_\rho - \sin(\varphi)\hat{\mathbf{e}}_\varphi] \\ &\quad + F_y[\sin(\varphi)\hat{\mathbf{e}}_\rho + \cos(\varphi)\hat{\mathbf{e}}_\varphi] + F_z\hat{\mathbf{e}}_z = \begin{pmatrix} F_x\cos(\varphi) + F_y\sin(\varphi) \\ -F_x\sin(\varphi) + F_y\cos(\varphi) \\ F_z \end{pmatrix}_p\end{aligned}$$

Cartesian and cylindrical polar components for the vector field $\mathbf{F} = (-\sin(\varphi), \cos(\varphi), 0)_c^T = (0, 1, 0)_p^T$ are shown in figures 1b) and 1c) respectively. Produce a similar figure for the vector field $\mathbf{F} = \mathbf{r} = (x, y, z)_c^T = (\rho, 0, z)_p^T$.

The most commonly used coordinate systems produce orthogonal right handed bases. This means that scalar and vector products between the basis vectors obey the familiar relations $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ and $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_k \epsilon_{ijk}$, where δ_{ij} is the Kronecker delta and ϵ_{ijk} the totally antisymmetric tensor. For cylindrical polar coordinates (ρ, φ, z) we explicitly have

$$\hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1 \quad \text{and} \quad \hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_\rho = 0 \,,$$

$$\hat{\mathbf{e}}_\rho \times \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_z = 0, \quad \hat{\mathbf{e}}_\rho \times \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_z \,, \quad \hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_\rho \quad \text{and} \quad \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_\varphi \,,$$

Vector Calculus

Partial derivatives

The partial derivatives with respect to the coordinates are found using the chain rule[2]

$$\partial_\rho = \cos(\varphi)\partial_x + \sin(\varphi)\partial_y \,, \qquad \partial_\varphi = -\rho\sin(\varphi)\partial_x + \rho\cos(\varphi)\partial_y \,, \quad \text{and} \quad \partial_z = \partial_z \,,$$

and

$$\partial_x = \cos(\varphi)\partial_\rho - \frac{\sin(\varphi)}{\rho}\partial_\varphi \,, \qquad \partial_y = \sin(\varphi)\partial_\rho + \frac{\cos(\varphi)}{\rho}\partial_\varphi \,, \quad \text{and} \quad \partial_z = \partial_z \,.$$

∇-operator

We now rewrite the ∇-operator by using these relations

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}_c = \hat{\mathbf{e}}_x\partial_x + \hat{\mathbf{e}}_y\partial_y + \hat{\mathbf{e}}_z\partial_z = \hat{\mathbf{e}}_\rho\partial_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho}\partial_\varphi + \hat{\mathbf{e}}_z\partial_z = \begin{pmatrix} \partial_\rho \\ \frac{1}{\rho}\partial_\varphi \\ \partial_z \end{pmatrix}_p$$

[1] This might seem obvious but needs to be revisited in relativity and has far reaching consequences for the physical nature of space.

[2] The calculation is identical to working out the basis vectors but replacing $\partial\mathbf{r}/\partial i \to \partial_i$. The partial derivatives can thus directly be read off from the relations between the basis vectors.

Gradient

The gradient of a scalar field U in cylindrical polar coordinates is now given by

$$\mathrm{grad}U = \nabla U = \begin{pmatrix} \frac{\partial U}{\partial \rho} \\ \frac{1}{\rho}\frac{\partial U}{\partial \varphi} \\ \frac{\partial U}{\partial z} \end{pmatrix}_p = \hat{\mathbf{e}}_\rho \partial_\rho U + \frac{\hat{\mathbf{e}}_\varphi}{\rho} \partial_\varphi U + \hat{\mathbf{e}}_z \partial_z U \,.$$

The expression for the ∇-operator in cylindrical polar components is thus indirectly given on the data-sheet[3]. There you will find an expression for ∇U and the del-operator is found by simply leaving out U in this expression.

Divergence

When working out the divergence we need to properly take into account that the basis vectors are not constant in general curvilinear coordinates. For cylindrical polar coordinates we have two nonzero derivatives

$$\partial_\varphi \hat{\mathbf{e}}_\varphi = -\cos(\varphi)\hat{\mathbf{e}}_x - \sin(\varphi)\hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_\rho \quad \text{and} \quad \partial_\varphi \hat{\mathbf{e}}_\rho = -\sin(\varphi)\hat{\mathbf{e}}_x + \cos(\varphi)\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\varphi \,.$$

The divergence will thus in general *not* be given by $\nabla \cdot \mathbf{F}(\mathbf{r}) = \sum_i \partial_i F_i(\mathbf{r})$ which is only true for an orthogonal coordinate system whose basis vectors are constant in space. Using the product rule we find

$$\begin{aligned} \nabla \cdot \mathbf{F} &= [\hat{\mathbf{e}}_\rho \partial_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho}\partial_\varphi + \hat{\mathbf{e}}_z \partial_z] \cdot [F_\rho \hat{\mathbf{e}}_\rho + F_\varphi \hat{\mathbf{e}}_\varphi + F_z \hat{\mathbf{e}}_z] = \partial_\rho F_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho} \cdot [\partial_\varphi(F_\rho \hat{\mathbf{e}}_\rho) + \partial_\varphi(F_\varphi \hat{\mathbf{e}}_\varphi)] + \partial_z F_z \\ &= \partial_\rho F_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho} \cdot [\hat{\mathbf{e}}_\rho \partial_\varphi F_\rho + F_\rho \hat{\mathbf{e}}_\varphi + \hat{\mathbf{e}}_\varphi \partial_\varphi F_\varphi - \hat{\mathbf{e}}_\rho F_\varphi] + \partial_z F_z = \frac{\partial_\rho(\rho F_\rho)}{\rho} + \frac{\partial_\varphi F_\varphi}{\rho} + \partial_z F_z \,. \end{aligned}$$

Rotation

The calculation is similar to working out the divergence

$$\begin{aligned} \nabla \times \mathbf{F} &= [\hat{\mathbf{e}}_\rho \partial_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho}\partial_\varphi + \hat{\mathbf{e}}_z \partial_z] \times [F_\rho \hat{\mathbf{e}}_\rho + F_\varphi \hat{\mathbf{e}}_\varphi + F_z \hat{\mathbf{e}}_z] = \hat{\mathbf{e}}_\rho \times [\hat{\mathbf{e}}_\varphi \partial_\rho F_\varphi + \hat{\mathbf{e}}_z \partial_\rho F_z] + \\ &\quad \frac{\hat{\mathbf{e}}_\varphi}{\rho} \times [\hat{\mathbf{e}}_\rho \partial_\varphi F_\rho - \hat{\mathbf{e}}_\rho F_\varphi + \hat{\mathbf{e}}_z \partial_\varphi F_z] + \hat{\mathbf{e}}_z \times [\hat{\mathbf{e}}_\rho \partial_z F_\rho + \hat{\mathbf{e}}_\varphi \partial_z F_\varphi] \\ &= \hat{\mathbf{e}}_\rho \left[\frac{\partial_\varphi F_z}{\rho} - \partial_z F_\varphi\right] + \hat{\mathbf{e}}_\varphi [-\partial_\rho F_z + \partial_z F_\rho] + \hat{\mathbf{e}}_z \left[\frac{\partial_\rho(\rho F_\varphi)}{\rho} - \frac{\partial_\varphi F_\rho}{\rho}\right] \,. \end{aligned}$$

Laplace operator

We obtain the Laplace operator by replacing $\mathbf{F} \to \nabla$ in the expression for the divergence

$$\Delta = \nabla \cdot \nabla = \frac{1}{\rho}\partial_\rho(\rho \partial_\rho) + \frac{1}{\rho^2}\partial_\varphi^2 + \partial_z^2 \,.$$

Example

A velocity field describing a vortex is given by $\mathbf{u} = A\hat{\mathbf{e}}_\varphi/\rho$ with A a constant. Its divergence and rotation are given by

$$\nabla \cdot \mathbf{u} = \left[\hat{\mathbf{e}}_\rho \partial_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho}\partial_\varphi + \hat{\mathbf{e}}_z \partial_z\right] \cdot \frac{A\hat{\mathbf{e}}_\varphi}{\rho} = 0 \,, \qquad \nabla \times \mathbf{u} = \left[\hat{\mathbf{e}}_\rho \partial_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho}\partial_\varphi + \hat{\mathbf{e}}_z \partial_z\right] \times \frac{A\hat{\mathbf{e}}_\varphi}{\rho} = \frac{A}{\rho^2}(\hat{\mathbf{e}}_z - \hat{\mathbf{e}}_z) = 0 \,.$$

The acceleration of a parcel of fluid is given by (this term appears in the Navier Stokes equation)

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{A}{\rho^2}\partial_\varphi \frac{A\hat{\mathbf{e}}_\varphi}{\rho} = -\frac{A}{\rho^3}\hat{\mathbf{e}}_\rho \,.$$

Work out these expressions using cartesian components.

[3] Also spherical polar coordinates can be found on the data sheet.

Summary

Cylindrical polar coordinates (ρ, φ, z)

- Relation to cartesian coordinates (care is required to obtain φ in the correct quadrant)

$$x = \rho\cos(\varphi)\,, \qquad y = \rho\sin(\varphi) \quad \text{and} \quad z = z$$
$$\rho = \sqrt{x^2+y^2}\,, \qquad \varphi = \arctan(y/x) \quad \text{and} \quad z = z$$

- Basis vectors $\hat{\mathbf{e}}_\rho = \partial_\rho \mathbf{r}$, $\rho\hat{\mathbf{e}}_\varphi = \partial_\varphi \mathbf{r}$, $\hat{\mathbf{e}}_z = \partial_z \mathbf{r}$

$$\hat{\mathbf{e}}_\rho = \cos(\varphi)\hat{\mathbf{e}}_x + \sin(\varphi)\hat{\mathbf{e}}_y\,, \qquad \hat{\mathbf{e}}_\varphi = -\sin(\varphi)\hat{\mathbf{e}}_x + \cos(\varphi)\hat{\mathbf{e}}_y \quad \text{and} \quad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$$
$$\hat{\mathbf{e}}_x = \cos(\varphi)\hat{\mathbf{e}}_\rho - \sin(\varphi)\hat{\mathbf{e}}_\varphi\,, \qquad \hat{\mathbf{e}}_y = \sin(\varphi)\hat{\mathbf{e}}_\rho + \cos(\varphi)\hat{\mathbf{e}}_\varphi \quad \text{and} \quad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z$$

- Non-zero derivatives of basis vectors

$$\partial_\varphi \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_\varphi\,, \qquad \partial_\varphi \hat{\mathbf{e}}_\varphi = -\hat{\mathbf{e}}_\rho$$

- ∇-operator

$$\nabla = \hat{\mathbf{e}}_\rho \partial_\rho + \frac{\hat{\mathbf{e}}_\varphi}{\rho}\partial_\varphi + \hat{\mathbf{e}}_z \partial_z$$

- Divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial_\rho(\rho F_\rho)}{\rho} + \frac{\partial_\varphi F_\varphi}{\rho} + \partial_z F_z$$

- Rotation

$$\nabla \times \mathbf{F}(\mathbf{r}) = \hat{\mathbf{e}}_\rho \left[\frac{\partial_\varphi F_z}{\rho} - \partial_z F_\varphi\right] + \hat{\mathbf{e}}_\varphi\left[-\partial_\rho F_z + \partial_z F_\rho\right] + \hat{\mathbf{e}}_z \left[\frac{\partial_\rho(\rho F_\varphi)}{\rho} - \frac{\partial_\varphi F_\rho}{\rho}\right]$$

- Laplace

$$\Delta = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{1}{\rho^2}\partial_\varphi^2 + \partial_z^2$$

Spherical polar coordinates (r, θ, φ)

- Relation to cartesian and cylindrical coordinates (care is required to obtain φ in the correct quadrant)

$$x = r\cos(\varphi)\sin(\theta)\,, \qquad y = r\sin(\varphi)\sin(\theta)\,, \qquad z = r\cos(\theta) \quad \text{and} \quad \rho = r\sin(\theta)$$
$$r = \sqrt{x^2+y^2+z^2}\,, \qquad \varphi = \arctan(y/x) \quad \text{and} \quad \theta = \arccos(z/r) = \arctan(\rho/z)$$

- Basis vectors $\hat{\mathbf{e}}_r = \partial_r \mathbf{r}$, $r\hat{\mathbf{e}}_\theta = \partial_\theta \mathbf{r}$, $r\sin(\theta)\hat{\mathbf{e}}_\varphi = \partial_\varphi \mathbf{r}$

$$\hat{\mathbf{e}}_r = \cos(\theta)\hat{\mathbf{e}}_z + \sin(\theta)\hat{\mathbf{e}}_\rho\,, \qquad \hat{\mathbf{e}}_\theta = -\sin(\theta)\hat{\mathbf{e}}_z + \cos(\theta)\hat{\mathbf{e}}_\rho \quad \text{and} \quad \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_\varphi$$
$$\hat{\mathbf{e}}_z = \cos(\theta)\hat{\mathbf{e}}_r - \sin(\theta)\hat{\mathbf{e}}_\theta\,, \qquad \hat{\mathbf{e}}_\rho = \sin(\theta)\hat{\mathbf{e}}_r + \cos(\theta)\hat{\mathbf{e}}_\theta \quad \text{and} \quad \hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_\varphi$$
$$\hat{\mathbf{e}}_x = \sin(\theta)\cos(\varphi)\hat{\mathbf{e}}_r + \cos(\theta)\cos(\varphi)\hat{\mathbf{e}}_\theta - \sin(\varphi)\hat{\mathbf{e}}_\varphi \quad \text{and} \quad \hat{\mathbf{e}}_y = \sin(\theta)\sin(\varphi)\hat{\mathbf{e}}_r + \cos(\theta)\sin(\varphi)\hat{\mathbf{e}}_\theta + \cos(\varphi)\hat{\mathbf{e}}_\varphi$$

- Non-zero derivatives of basis vectors

$$\partial_\theta \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta\,, \qquad \partial_\theta \hat{\mathbf{e}}_\theta = -\hat{\mathbf{e}}_r\,, \qquad \partial_\varphi \hat{\mathbf{e}}_r = \sin(\theta)\hat{\mathbf{e}}_\varphi\,, \qquad \partial_\varphi \hat{\mathbf{e}}_\theta = \cos(\theta)\hat{\mathbf{e}}_\varphi \quad \text{and} \quad \partial_\varphi \hat{\mathbf{e}}_\varphi = -\hat{\mathbf{e}}_\rho$$

- ∇-operator

$$\nabla = \hat{\mathbf{e}}_r \partial_r + \frac{\hat{\mathbf{e}}_\theta}{r}\partial_\theta + \frac{\hat{\mathbf{e}}_\varphi}{r\sin(\theta)}\partial_\varphi$$

- Divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial_r(r^2 F_r)}{r^2} + \frac{\partial_\theta(\sin(\theta)F_\theta)}{r\sin(\theta)} + \frac{\partial_\varphi F_\varphi}{r\sin(\theta)}$$

- Rotation

$$\nabla \times \mathbf{F} = \frac{\hat{\mathbf{e}}_r}{r\sin(\theta)}\left[\partial_\theta(F_\varphi \sin(\theta)) - \partial_\varphi F_\theta\right] + \frac{\hat{\mathbf{e}}_\theta}{r}\left[-\partial_r(rF_\varphi) + \frac{\partial_\varphi F_r}{\sin(\theta)}\right] + \frac{\hat{\mathbf{e}}_\varphi}{r}\left[\partial_r(rF_\theta) - \partial_\theta F_r\right]$$

- Laplace

$$\Delta = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta) + \frac{1}{r^2\sin^2(\theta)}\partial_\varphi^2$$