Chaos, Chance & Predictability

Problem Set 1: Deterministic Systems

1. Given $\dot{x} = \sin x$,

(a) Find all the fixed points of the flow,

(b) At which points x does the flow have the greatest velocity to the right?

(c) Find the flow's acceleration $\ddot{x}$ as a function of x and show where this acceleration has its positive maximum.

2. Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails (because $f'(x^*) = 0$), use a graphical argument to determine the stability:

(a) $\dot{x} = x(1-x)$,

(b) $\dot{x} = \tan x$,

(c) $\dot{x} = 1 - e^{-x^2}$.

3. Prove that periodic solutions are impossible for vector fields on a line. Suppose on the contrary that $x(t)$ is a nontrivial periodic solution (i.e. that $x(t) = x(t+T)$ for some time $T > 0$, and $x(t) \neq x(t+s)$ for all $0 < s < T$). Derive a contradiction by considering

$$\int_t^{t+T} f(x) \frac{dx}{dt} dt \cdot$$

4. For each of the following vector fields, plot the potential function $V(x)$ and identify all the equilibrium points and their stability:

(a) $\dot{x} = x(1-x)$,

(b) $\dot{x} = \sin x$,

(c) $\dot{x} = -\sinh x$,

(d) $\dot{x} = r + x - x^3$ for several values of r.

5. In each of the following, find the values of r at which bifurcations occur, and classify them as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork. Sketch the bifurcation diagram of fixed points x^* vs r.

(a) $\dot{x} = r - 3x^2$,

(b) $\dot{x} = rx - x/(1+x)$,

(c) $\dot{x} = 5 - re^{-x^2}$,

(d) $\dot{x} = rx - x/(1+x^2)$.

6. Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

(a) Write the system as $\dot{\mathbf{x}} = \underline{\mathbf{A}}\mathbf{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of $\underline{\mathbf{A}}$.

(b) Find the general solution of the system.

(c) Classify the fixed point at the origin.

(d) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

7. Plot a phase portrait and classify the fixed points of each of the following linear systems:

(a) $\dot{x} = y, \quad \dot{y} = -2x - 3y,$

(b) $\dot{x} = 3x - 4y, \quad \dot{y} = x - y,$

(c) $\dot{x} = 5x + 2y, \quad \dot{y} = -17x - 5y,$

(d) $\dot{x} = 4x - 3y, \quad \dot{y} = 8x - 6y$

8. For each of the following nonlinear systems, find the fixed points, classify them, sketch the neighbouring trajectories and try to fill in the rest of the phase portrait.

(a) $\dot{x} = x - y, \quad \dot{y} = x^2 - 4,$

(b) $\dot{x} = 1 + y - e^{-x}, \quad \dot{y} = x^3 - y,$

(c) $\dot{x} = \sin y, \quad \dot{y} = x^3 - y$

9. Variant on the Lotka-Volterra predator-prey model (Rabbits vs foxes?). This may be written

$$\begin{aligned} \dot{R} &= aR - bRF \\ \dot{F} &= -cF + dRF, \end{aligned}$$

where $R(t)$ is the number of rabbits, $F(t)$ is the number of foxes, and $a, b, c, d > 0$ are constants.

(a) Discuss the biological meaning of each term in the model (commenting on unrealistic assumptions).

(b) By rescaling the variables R, F and time t in the form $nx = R; my = F$ and $r\tau = t$ to make them non-dimensional, show that the model can be recast in dimensionless form as $x' = x(1 - y)$, $y' = \mu y(x - 1)$ (where $['] = d/d\tau$). Find expressions for the scale parameters n, m, τ and μ in terms of a, b, c and d.

(c) Show that a conserved quantity for the dimensionless system exists of the form $\Lambda = Ax + By + C \ln x + D \ln y$, and find expressions for the constants A, B, C and D consistent with $B = 1$.

(d) Show that the model predicts cycles in the populations of both species for almost all initial conditions. Comment on what kind of cycles these are?

10. The equation $\ddot{\theta} + \sin\theta = \gamma$ represents the dynamics of an undamped pendulum driven by a constant torque (or possibly an undamped Josephson Junction driven by a constant bias current).

(a) Find all the equilibrium points and classify them as γ is varied.

(b) Sketch the nullclines and the vector field.

(c) Is the system conservative? If so, find a conserved quantity.

(d) Sketch the phase portrait on the plane as γ varies.

(e) Find the approximate frequency of small oscillations about any centres in the phase portrait.

11. Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in (μ, a) space at which Hopf bifurcations occur.

12. The Lorenz equations provide a simple model of fluid convection, and are given by:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\cdot\end{aligned}$$

(a) Identify the nonlinearities in the equations and explain the physical significance of the parameters r, σ and b.

(b) Calculate the fixed points of the system and comment on their physical significance. Establish stability criteria for the origin, and state how the other fixed points lose stability.

(c) Sketch the chaotic Lorenz attractor and show how it relates to the respective stability of the fixed points.

(d) Show that the Lorenz equations are dissipative, and relate the result to the attractors for the system.

13. The quadratic iterative map is given by:

$$x_{n+1} = 4\lambda x_n(1 - x_n)$$

where λ is a parameter and the map is restricted to the range $0 \leq x_n \leq 1$.

(a) Find the fixed points of the map for $0 \leq \lambda \leq 1/2$.

(b) Establish a stability criterion for the map, and hence find the range of stability of the fixed points. Sketch diagrams to illustrate your answer.

14. Consider the so-called 'tent map':

$$\begin{aligned} x_{n+1} &= 2rx_n \quad \text{for } 0 \le x_n \le 1/2 \\ x_{n+1} &= r(2-2x_n) \quad \text{for } 1/2 \le x_n \le 1, \end{aligned}$$

where r is a parameter.

(a) Sketch the form of the map.

(b) Find the fixed points of the map and determine their stability as a function of r. Illustrate your answer with sketches.

(c) Compare and contrast the routes to chaos in the quadratic and tent maps. Discuss whether the results for the tent map contradict the Feigenbaum conjecture?

[End of paper]