Wave Optics

Propagation, interference and diffraction of waves

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Paul Ewart’s lecture notes and problem sets:

https://www2.physics.ox.ac.uk/research/combustion-physics-and-non-linear-optics/teaching

Wave Optics – Literature

- Brooker, Modern Classical Optics
- Hecht, Optics
- Klein and Furtak, Optics
- Smith, King & Wilkins, Optics and Photonics
- Born and Wolf, Principles of Optics

Wave Optics – Outline

- What’s it all about?
- Revision of geometrical optics
- Propagation of waves
- Fourier methods
  - Fresnel-Kirchhoff integral, theory of imaging
- Diffraction-based optical instruments
  - 2-slit, grating, Michelson and Fabry-Perot Interferometer
- Dielectric surfaces and boundaries
  - multilayer (anti)reflection coatings
- Polarized Light

What’s it all about?

- Imaging
- Visualization (projection, lithography)
- Spectroscopy
- Matter-wave propagation & imaging
- Lasers and applications
- Modern devices
  (opto-electronics, display technology, optical coatings, telecommunication, consumer electronics)
What’s it all about?

Astronomical observatory, Hawaii, 4200m above sea level.

What’s it all about?

Hubble space telescope, 2.4 m mirror

What’s it all about?

Optical Microscope

fruit fly
What’s it all about?

CD/DVD player optical pickup system

What’s it all about?

cutting & welding

photo lithography

What’s it all about?

Coherent Light ➔ Laser Physics

- spectroscopy
- metrology (clocks)
- quantum optics
- quantum computing
- laser nuclear ignition
- medical applications
- engineering
- telecommunication

Geometrical Optics – Revision

- Fermat’s principle (shortest path)
- reflection & refraction
- spherical & thin lenses
- paraxial approximation
- lensmaker’s formula
- combining lenses
- principal planes
- optical instruments
- Aperture and field stops
- Pinhole camera ➔ wave optics
Revision of Geometrical Optics

- Light rays → straight lines
- Reflection: \( \Theta = \Phi \)

Refraction

Snell's law
\[ n_1 \sin \Theta_1 = n_2 \sin \Theta_2 \]

Speed of light \( V = \frac{c}{n} \) 
\((n = \text{Refractive index})\)
Fermat's Principle

Light takes 'shortest' optical path from A to B

\[ \text{OPL} = n \times \text{Real path length} \]

\[ \text{OPL} = \sqrt{h^2 + x^2} + \sqrt{h^2 + (L-x)^2} \]

\[ \frac{d}{dx}(\text{OPL}) = 0 \quad \therefore \quad x = \frac{L}{2} \quad \therefore \quad \theta = \phi \]

→ Snell's law in similar manner
Simple Imaging → lenses

1. Spherical surface

\[ \frac{h}{R} = \sin \Theta = n \sin \phi \approx n (\Theta - \frac{h}{R}) = n \left( \frac{h}{R} - \frac{h}{f} \right) \]

\[ \Rightarrow \quad \frac{n}{f} = (n-1) \frac{1}{R} \quad \text{Single sph. surface} \]

Node: Symmetric →

* focal points on a sphere
* image from sphere to sphere
also: \[ \frac{1}{u} + \frac{1}{v} = \frac{1}{f} = (n-1) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

**Paraxial approximation**
- small angles: \( \theta \approx \sin \theta \) etc.
- neglect curvature of focal or image planes

**Thin lens**
- neglect propagation inside
- two curved surfaces

**Lensmaker's equation**
basic lens properties

1. central rays not refracted
2. rays parallel to optic axis go through focal point on other side
3. parallel bundles of rays travelling under angle 2 on one side all go through the same point in the focal plane on the other side:

\[ y = \frac{2}{d} \cdot f \]
Lens thickness from Fermat’s principle

(Form follows function)

\[ \overrightarrow{\text{Fermat: shortest OPL from } A \text{ to } B} \]

\[ \overrightarrow{\text{B image of } A \text{ if all OPL's of same length}} \]

\[ \rightarrow \text{ lens thickness } d(h) = d_0 - \Delta(h) \]

\[ \text{OPL}(h) = \sqrt{h^2 + u^2} + \sqrt{h^2 + v^2} + (n-1)(d_0 - \Delta(h)) \]

\[ = \text{OPL}(0) = u + v + (n-1)d_0 \]

= const. \( (h \text{ independent}) \)
\[ \Delta(h) \cdot (n-1) = \sqrt{h^2u^2 + h^2v^2} - u - v \]

\[
\frac{1}{2R} (n-1) \approx \frac{h^2}{2u} + \frac{h^2}{2v}
\]

**Oracle**: plano-convex lens

\[ \Delta = h \left( \sqrt{h^2 + R^2} - R \right) = \frac{h^2}{2R} \]

**Therefore**: \[ \frac{1}{R} (n-1) = \frac{1}{u} + \frac{1}{v} \]

**Widespread use**: [microscopes, telescopes, cameras + projectors, magnifying glasses, eyepiece, etc.]

\[ \text{Imaging - Magnification - Observation - Lithography} \]
Angle subtended by virtual image:

\[ \theta = \frac{h}{u} = \frac{h'}{D} \]

\[ \frac{1}{\theta} = \frac{1}{u} + \frac{1}{v} = \frac{1}{u} + \frac{1}{-D} \]

\[ \frac{1}{D} + \frac{1}{D} = \frac{1}{u} \rightarrow \theta = \frac{h}{\theta} + \frac{h'}{D} \]

Naked eye -- object at D:

Angle \( \theta' = \frac{h'}{D} \)

Magnification:

\[ M = \frac{\theta}{\theta'} = \frac{D}{u} + 1 \]
**Fermat’s Principle**

Light propagating between two points follows a path, or paths, for which the time taken is an extremum (minimum).

- Ignoring the wave nature of light
- Basic theory for optical instruments

**Geometrical Optics – Revision**

- Geometric focussing with spherical surfaces
- Parallel bundles
- Image sphere
- Object sphere
- Image sphere

**Thin lens formula**

\[
\frac{1}{u} + \frac{1}{v} = \frac{1}{f}
\]

**Geometrical Optics – Instruments**

- Principal planes
- Location of equivalent thin lens
- First Principal Plane
- Back Focal Plane
Thin lens equation applies with \( u \) and \( v \) measured from the two principal planes.

Geometrical Optics – Instruments

Principal planes

Front Focal Plane
Second Principal Plane

Objective magnification = \( v/u \)
Eyepiece magnifies real image of object

Angular magnification = \( \beta/\alpha = f_0/f_E \)
Image brightness

light collected \sim D^2 \\
(\text{less area })

object area \sim d^2

Image area \sim (2df)^2

\therefore \text{brightness} \sim

\frac{\text{lens area}}{\text{image area}} \sim \frac{D^2}{(2df)^2} \sim \left(\frac{f}{D}\right)^{-2} = (f\text{-no.})^{-2}

f\text{-no.} = \frac{\text{local length}}{\text{pupil diameter}}
Combining two lenses

\[ u = \frac{1}{f_1} \quad v = \frac{1}{f_2} \]

\[ \frac{1}{u} + \frac{1}{v} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f} \text{ if lenses are close} \]

\( \frac{1}{f} \) is the power of the lens, measured in dioptries: \([\text{dioptries}] = [\text{m}^{-1}]\)
Ray transfer matrices

Ray vector \( \left( \begin{array}{c} y \\ \theta \end{array} \right) \) ← dist. from axis
← angle to axis

\[
\left( \begin{array}{c} y_2 \\ \theta_2 \end{array} \right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left( \begin{array}{c} y_1 \\ \theta_1 \end{array} \right)
\]

Transfer

\[
S_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}
\]

\[
y_2 = y_1 + \theta_1 d \\
\theta_2 = \theta_1
\]

\[
S_l = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}
\]

\[
y_2 = y_1 \\
\theta_2 = \theta_1 - \frac{y_1}{f}
\]

Ray transfer through optical system

\[
\left( \begin{array}{c} y_e \\ \theta_e \end{array} \right) = S_{d_4} S_{d_3} S_{d_2} S_{d_1} S_{d_0} \left( \begin{array}{c} y_s \\ \theta_s \end{array} \right)
\]

System transfer
1.3.4 Telescope (Galilean)

\[ \text{angular magnification} = \frac{\beta}{\alpha} = \frac{f_o}{f_E} \]

Figure 1.6

1.3.5 Telescope (Newtonian)

Figure 1.7

1.3.6 Compound Microscope

Figure 1.8

Apertures and Field Stops

\[ \text{f\,no.} = \frac{\text{focal length}}{\text{entrance pupil diameter}} \]

The Wave Nature of Light

- Maxwell’s equations \( \rightarrow \) waves
- equivalence to matter waves
- plane and spherical waves
- energy flow / intensity
- Huygen’s principle

Camera Obscura

optimum pinhole size
contradicts expectations from geometrical optics

from Hecht, Optics
Wave Nature of Light

Maxwell with $\mathbf{B} = 0$ and $\mathbf{J} = 0$

\[ \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{H} = \frac{1}{\varepsilon} \frac{\partial \mathbf{D}}{\partial t} \]
\[ \nabla \cdot \mathbf{D} = 0 \quad \nabla \times \mathbf{E} = -\frac{1}{\varepsilon} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} \]
\[ \mathbf{B} = \mu_0 \mu_r \mathbf{B} \]
\[ -\nabla \times \nabla \times \mathbf{E} = \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2 \mathbf{E}}{\partial t^2} \]
\[ -\nabla(\nabla \cdot \mathbf{E}) + \nabla \cdot (\nabla \mathbf{E}) = \frac{1}{c^2} \eta^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

= 0

\[ \nabla^2 \mathbf{E} = \left( \frac{\eta}{c} \right)^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

→ Wave eqn. in each component of $\mathbf{E}$ (or $\mathbf{H}$)

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \text{and} \quad \eta = \sqrt{\mu_r \varepsilon_r} = \sqrt{\varepsilon_r} \]
Ansatz: $E = E_0 e^{-i\omega t}$

(or $H = H_0 e^{-i\omega t}$)

Note: Real components $= \frac{1}{2} (E + E^*)$

For each $E$ component:

$\Delta E = \left( \frac{n}{\lambda} \right)^2 \frac{\partial^2}{\partial x^2} E = -\omega^2 \left( \frac{n}{\lambda} \right)^2 E$

$\uparrow$

$(\Delta + n^2 k_0^2) E = 0$

or $(\Delta + \lambda^2) E = 0$

Wavevector $k = \frac{2\pi}{\lambda} = n k_0 = n \frac{2\pi}{\lambda_0} = n \frac{\omega}{c}$

Note: This also holds for matter waves.
\[ \nabla^2 \Psi = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \Psi = E \Psi \]

\[ L \to \left( \Delta + 2m \frac{E - V}{\hbar^2} \right) \Psi = 0 \]

with \[ 2m E_{\text{kinetic}} = p^2 = \hbar^2 k^2 \]

\[ L \to \left( \Delta + k^2 \right) \Psi = 0 \]

- Looks familiar
- \( \hbar^2 k^2 = 2m \left( E - V \right) \) for matter waves

Back to light \( (\Delta + \hbar^2) u = 0 \)

\( u = \) general amplitude \( \sim E \) or \( \hbar \)

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
Assume \( u(x,t) = u_0 \cdot e^{i(k \cdot x - \omega t)} \)

with \( |k| = nk_z = n \frac{\omega}{c} = \frac{\omega}{\nu_p} \)

\( (\nu_p = \frac{c}{n} = \text{phase velocity}) \)

\[ \rightarrow \text{solve the wave equation} \]

\[ \rightarrow \text{Plane Wave} \]

propagating with \( \nu_p = \)

![Diagram showing a plane wave with wavefronts perpendicular to \( k \).]

defined by planes of constant phase

\[ \frac{d}{dt}(k \cdot x - \omega t) = 0 \]
\[ \text{Observe assume } k = \left( \begin{array}{c} 0 \\ \kappa \end{array} \right) \]

\[ \Rightarrow k \frac{dz}{dt} = \omega \]

\[ \Rightarrow z = t \frac{\omega}{k} + \text{const.} = t V_p \text{ const.} \]

\( \omega \) wavefronts propagate with \( V_p \)

Field direction?  

\[ E(x,t) = E_0 e^{-i(kx - \omega t)} \]

--- into Maxwell's equations:

\[ \omega B = k \times E \quad \text{and} \quad -\omega D = k \times H \]

\[ k \cdot B = 0 \quad \text{and} \quad k \cdot D = 0 \]

\[ \Rightarrow \quad \boxed{B \perp k \quad D \parallel k} \quad \boxed{B \perp E \quad D \parallel H} \]

\[ \Rightarrow \text{EM waves mostly transverse!} \]
Energy density + Intensity

\[ S_{EM} = \frac{1}{2} (E \cdot D + B \cdot H) \]

Energy flow \rightarrow Poynting vector

\[ \mathbf{S} = \mathbf{E} \times \mathbf{H} \parallel \mathbf{k} \]

\[ [S] = \text{W/m}^2 \]

(Henry Poynting, 1884)

Intensity \[ I = \langle S \rangle_t \]

\[ = \frac{1}{2} \varepsilon_0 \varepsilon_r E_0^2 \sqrt{\mu_r \mu_0} \approx \frac{1}{2} \varepsilon_0 \varepsilon_r E_0^2 \frac{c}{n} \]

\[ \mu_r = 1 \quad \Rightarrow \quad S_{11} N_p \]
Other Solutions?

\[ \Delta = \frac{1}{r^2} \frac{d^2}{dr^2} r + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

This yields

\[ u(r) = \frac{u_0}{r} e^{\pm ikr - i\omega t} \]

\[ \pm ikr = \text{out}\text{-}or\text{ ingoing} \]

- Huygen's wavelets
Maxwell’s Equations

\[ \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{H} = \frac{d}{dt} \vec{D} \\
\nabla \cdot \vec{D} = 0 \quad \nabla \times \vec{E} = -\frac{d}{dt} \vec{B} \]

\[ \vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \quad \vec{B} = \mu_r \mu_0 \vec{H} \]

linear isotropic medium

Huygen’s Principle

Every point on a wave front can be considered as a source of secondary spherical waves

\[ u(\vec{R}) \propto \int u(\vec{r}) e^{i k |\vec{R} - \vec{r}|} dS \]

Plane and Spherical Waves

(\Delta + k^2)u = 0 \quad u \rightarrow \text{amplitude of } E, H, \psi \ldots

plane wave

\[ u(\vec{r}, t) = u_0 e^{i(k\vec{r} - \omega t)} \]

spherical wave

\[ u(\vec{r}, t) = u_0 e^{i(k\vec{r} - \omega t)}/r \]

\[ n = \sqrt{\varepsilon_r \mu_r} \quad c = 1/\sqrt{\varepsilon_0 \mu_0} \quad v_p = c/n = \omega/k = \nu \lambda \quad \vec{S} = \vec{E} \times \vec{H} \]

Interference of Waves

Huygens’ wavelet
Huygens' Principle

- Assume \( u_0(z) \) is known in an optical plane.
- \( u(R) \) is the superposition of spherical wavelets emanating from all points of the plane.

\[
u(R) = \int_{S} \frac{u(R)}{|R - \xi|} e^{-ik|R - \xi|} dS
\]

Special case: Surface \( S @ z = 0 \)

- \( \xi = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) \( R = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \)

\[
u(R) = \int \frac{u(x, y, 0)}{\sqrt{(x' - x)^2 + (y' - y)^2 + z'^2}} e^{ik\sqrt{s}} dS
\]

Fresnel - Kirchhoff integral