

Since light does not propagate (far) through good conductors such as gold and copper try ansatz of decaying wave

$$\vec{E}(z, t) = E_0 e^{-kz} e^{i(\omega t - kz)} \hat{x}$$

\hat{x} -polarized plane wave travelling in $+z$ dir'n

decaying 'envelope' factor (note no "i") — Can think of k as imaginary part of K

substitute this into good conductor eqn (*)

$$\text{LHS: } \nabla^2 \vec{E} = (-K - ik)^2 \vec{E}$$

$$\text{RHS: } \mu_r \mu_0 \sigma \frac{\partial \vec{E}}{\partial t} = i \mu_r \mu_0 \sigma \omega \vec{E}$$

\Rightarrow the dispersion relation (ie, rel'n between ω and real and imaginary parts of k)

$$(K + ik)^2 = i \mu_r \mu_0 \sigma \omega$$

Equate real and imaginary parts of this

$$\Rightarrow K^2 - k^2 = 0 \quad (\text{real part})$$

$$2iKk = i \mu_r \mu_0 \sigma \omega \quad (\text{imag. part})$$

$$\Rightarrow K = \pm k$$

physically must select
 $+k$ solution so that
 envelope factor is e^{-Kz}
 (ie decaying as z increases)

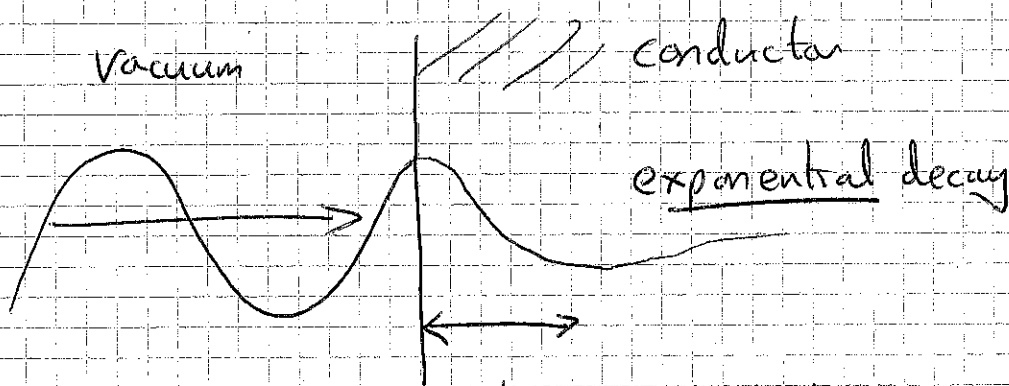
and

$$2K \cdot k = 2K^2 = \mu_r \mu_0 \sigma \omega$$

$$\Rightarrow K = \sqrt{\frac{\mu_r \mu_0 \sigma \omega}{2}}$$

again must
 select sign
 such that
 wave decays

Physically wave penetrates only a small distance into conductor



For metals

Visible freq: $\delta \sim \text{nm}$

microwave $\delta \sim \mu\text{m}$

radio $\delta \sim \text{mm}$

typical depth of penetration (distance for e^{-1} fall off)
 = "skin depth"

$$\delta \equiv 1/K = \sqrt{2/\mu_r \mu_0 \sigma \omega}$$

Real and Imaginary Refractive Indices (and $\epsilon_r(\omega)$)

The example of a conductor illustrates a very important point

Extremely useful to think in terms of a complex refractive index

For usual wave not undergoing absorption or amplification

have $k = \frac{n\omega}{c}$ which is purely real
↑
wavenumber

and associated wave solution

$$e^{i(\omega t - \frac{n\omega}{c} z)}$$

If let n be complex

$$n = n_R - i n_I$$

n_R } real numbers
 n_I } (actually functions
of ω in general)

NOTE: sign is annoying
convention

then solution is

$$e^{i\left(\omega t - \frac{n_R \omega}{c} z\right) - i(-i) \frac{n_I \omega z}{c}}$$

$$= e^{i\left(\omega t - \frac{n_R \omega}{c} z\right) - \frac{n_I \omega z}{c}}$$

n_R characterises the
group and phase
velocities of
propagation

n_I characterises the
absorption/amplification
of wave as it passes
through medium

[previous annoying sign is
so that $n_I > 0$ gives
'usual' case of absorption.

However in active materials,
where atoms are not in
their ground states, can
have amplification $n_I < 0$]

From now on subscripts 'R' and 'I' will always refer to
real and (-) imaginary parts of a quantity

By our earlier discussion of EM waves in media

$$n = \frac{c}{v_{\text{phase}}} = \sqrt{\epsilon_r \mu_r}$$

Most cases of physical interest for waves have $\mu_r \approx 1$ so

$$n(\omega) = \sqrt{\epsilon(\omega)}$$

now emphasising
freq. dep.

In this formula both
 $n(\omega)$ and $\epsilon(\omega)$ complex
functions of ω

drop subscript 'r' so
don't confuse with real and
imag. parts of $\epsilon(\omega)$

Can compute $\epsilon(\omega)$ by using definition of $\epsilon(\omega)$ in
terms of polarization density \vec{P} of medium

$$\epsilon(\omega) \epsilon_0 \vec{E} \equiv \epsilon_0 \vec{E} + \vec{P}$$

$e^{i\omega t}$ varying also $e^{i\omega t}$ varying

$$\Rightarrow \vec{P} = (\epsilon(\omega) - 1) \epsilon_0 \vec{E}$$

So if can compute \vec{P} for a medium when responding
to a monochromatic \vec{E} can get formula for $\epsilon(\omega)$

NOTE: Since $\epsilon(\omega)$ and $n(\omega)$ are functions of ω we will get
dispersion with $v_{\text{group}} \neq v_{\text{phase}}$

$\epsilon(\omega)$ for low density dielectrics

Really $\epsilon(\omega)$ needs to be computed using QM as the polarizability of atoms/molecules as a function of ω is a quantum property depending on energy levels and associated eigenfunctions. However, suitably interpreted, a simple classical model gives a remarkably good description for low density dielectrics (eg, low density gases)

Assume the bound electrons can be modelled as classical damped harmonic oscillators

$$m \ddot{x} + m \gamma \dot{x} + m \omega_0^2 x = -e E_0 e^{i\omega t}$$

x = displacement of e^- from equilibrium
so this is $(F \Rightarrow) ma$ term

a damping term

- can exist due to collisions/atom- e^- interaction

and must be present due to

radiation from accelerating charge which takes off energy!

restoring force on bound electron

ω_0 = resonant frequency

note only the \vec{E} field - the \vec{B} dep't part of Lorentz force is subdominant in non-relativistic case

To solve this assume $x(t) = x_0 e^{i\omega t}$

$$\Rightarrow m(-\omega^2 x_0 + i\omega \gamma x_0 + \omega_0^2 x_0) e^{i\omega t} = -e E_0 e^{i\omega t}$$

$$\Rightarrow x_0 = \left(\frac{-e E_0}{m} \right) \frac{1}{\omega_0^2 - \omega^2 + i\omega \gamma}$$

With this displacement there is an associated dipole moment at atom

$$p = -e x_0 = \frac{e^2 E_0}{m} \frac{1}{\omega_0^2 - \omega^2 + i\omega \gamma}$$

Now to get total polarization density \vec{P} take

N = number-density of atoms (or molecules if appropriate)

$$\Rightarrow \vec{P} = \frac{e^2 N E_0}{m} \frac{1}{\omega_0^2 - \omega^2 + i\omega \gamma}$$

total pol'n
density

$$= \epsilon_0 (\epsilon(\omega) - 1) E_0 \quad \text{by earlier formula}$$

$$\Rightarrow \boxed{\epsilon(\omega) = 1 + \frac{e^2 N}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\omega \gamma}}$$

classical result for $\epsilon(\omega)$

Quantum mechanically a single atom or molecule can have many resonant frequencies $\omega_0 = \frac{\Delta E}{\hbar}$ for its electronic excitations where ΔE are all the possible $E_e - E_m$ energy differences ~~possible~~ between energy eigenstates

Let's label all the possible resonant frequencies by 'j'

$$\begin{aligned} \text{set of possible frequencies} \\ \text{of excitation of atom} &= \{ \omega_j \} \\ & \quad j=1, 2, 3, \dots \end{aligned}$$

Associated with each of these is a damping term γ_j (which in general varies from excitation mode to excitation mode)

Finally the "1" in the numerator of the classical expression $1/(\omega_0^2 - \omega^2 + i\omega\gamma)$ gets replaced by an "oscillator strength" f_j (a pure number) which ^{roughly} expresses the contribution of ^{an} excitation (note f_j is not equal to the probability of an excitation - for example for active materials with atoms prepared in excited states $f_j < 0$ is possible for some j's)

So summing over all the QM resonant frequencies we get correct QM form of $\epsilon(\omega)$

$$\epsilon(\omega) = 1 + \frac{e^2 N}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 + i\omega\delta_j}$$

Comments:

- ① For low densities (ie small N) and not too small δ_j 's (so damping not zero, so no 'perfect' resonant behaviour) the 2nd term in above eqn is small compared to 1.

Since $n(\omega) = \sqrt{\epsilon(\omega)}$ we can expand out using Taylor expansion to find

$$n(\omega) \approx \frac{e^2 N}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 + i\omega\delta_j}$$

This has both real and imaginary parts

— will discuss physics of this soon.

- ② ω_j , f_j and δ_j have to be calculated from QM. In our theory we will just take these values as given.