EM Waves in Materials

Expressed in terms of $\vec{D}$ and $\vec{H}$ Maxwell's eqns in a medium are

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

To solve these eqns we need to suplement them with the appropriate relations between $\vec{D}, \vec{H}$ and $\vec{E}, \vec{B}$

(1) For a linear, isotropic, homogeneous dielectric

$$\vec{B} = \mu_r \mu_0 \vec{H}$$

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}$$

with μ_r, ϵ_r indep't of position (and as scalars, indep't of direction too)

Thus in absence of free currents and charges Maxwell's eqns become identical in form to the vacuum Maxwell eqns but with replacements

$$\epsilon_0 \longrightarrow \epsilon_r \epsilon_0$$

$$\mu_0 \longrightarrow \mu_r \mu_0$$

Therefore can immediately read off properties of waves in a linear, isotropic, homogeneous dielectric

$$v = \frac{1}{\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0}}$$

$$n = \text{refractive index} = \frac{c}{v}$$

$$= \sqrt{\mu_r \epsilon_r}$$

<u>Comment</u>: Remember that ϵ_r and μ_r are <u>functions</u> of frequency, ω (and other things too like T)

This will be important soon...

Can also calculate from Maxwell's eqns the ratio of $|E|/|B|$ for a wave

$$\left| \frac{E_0}{B_0} \right| = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0}}$$

More useful in most circumstances is the ratio $|E_0 / H_0|$ which is known as the impedance (units Ohms) Z

$$Z = \left| \frac{E_0}{H_0} \right| = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}}$$

— this determines reflection/transmission at boundaries between materials

Free space (vacuum) also has an impedance

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$$

(2) Plane waves in conductors

no free charge (except possibly at boundary - ignore this case)

Now have for a linear, homogeneous, isotropic material

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}$$

$$\vec{B} = \mu_r \mu_0 \vec{H}$$

and

$$\vec{J} = \underset{\text{conductivity}}{\sigma} \vec{E}$$

Comment: Ohm's law is quite surprising if you think about it - would naively think the force due to $\vec{E}$ should give rise to a constant acceleration of charges, not a constant velocity (giving a constant current).

I'll leave the microscopic understanding of Ohm's Law for another course...

Thus Maxwell's eqns for a conductor are

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t$$

$$\vec{\nabla} \times \vec{B} = \mu_r \mu_0 \sigma \vec{E} + \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

However most conductors can be classified as either 'good' or 'poor' conductors depending on a ratio of timescales:

(a) Timescale associated to EM wave of freq ω

$$t_{wave} \sim 1/\omega$$

(b) Timescale associated to rate of change of charge distributions

Can find this by starting with a non-zero charge dist'n at $t=0$, $\rho_f(0)$, and asking how quickly it decays to zero.

Cons of charge $\Rightarrow$ $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

now use $\vec{J} = \sigma \vec{E}$

$$\Rightarrow \sigma \vec{\nabla} \cdot \vec{E} = -\frac{\partial \rho}{\partial t}$$

now use Gauss' Law

$$\Rightarrow \frac{\sigma \rho}{\epsilon_r \epsilon_0} = -\frac{\partial \rho}{\partial t}$$

This has solution

$$\rho(t) = \rho(0)\, e^{-\sigma t/\epsilon_r \epsilon_0}$$

$\Rightarrow$ characteristic timescale

$$t_{charge} = \frac{\epsilon_r \epsilon_0}{\sigma}$$

if σ large then t_{charge} small

This allows us to define 'good' vs 'poor' conductors

<u>GOOD</u>

$t_{charge} \ll t_{wave}$ (so charge dist'n can keep up with fields of EM wave)

$$\Rightarrow \frac{\epsilon_r \epsilon_0}{\sigma} \ll \frac{1}{\omega}$$

$$\Rightarrow \boxed{\frac{\sigma}{\omega \epsilon_r \epsilon_0} \gg 1}$$

POOR

$t_{charge} \gg t_{wave}$ (charge dist'n essentially static on timescale of wave)

$$\Longrightarrow \boxed{\frac{\sigma}{\omega \epsilon_r \epsilon_0} \ll 1}$$

Comments: (1) This distinction is frequency dep't both because of explicit $1/\omega$ and because $\epsilon_r(\omega)$, $\sigma(\omega)$ are functions of ω in general.

The general trend is that as one goes to higher frequencies conductors become poorer but this is not an absolute statement

(2) The condition for poor or good conductors is the same as the one that determines which of the two terms on the RHS of the 4th Maxwell eqn dominates

↳ see over

Specifically

$$\vec{\nabla} \times \vec{B} = \mu_r \mu_0 \sigma \vec{E} + \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Ampere term
(the conduction current term)

Maxwell term
(the displacement current term)

for a $\vec{E} \sim e^{i\omega t}$ find

$$\frac{\text{conduction term}}{\text{displacement term}} \sim \frac{\sigma \mu_r \mu_0}{\mu_r \mu_0 \epsilon_r \epsilon_0 \omega}$$

$$\sim \frac{\sigma}{\epsilon_r \epsilon_0 \omega}$$

$\Longrightarrow$
- good conductor $\Rightarrow$ conduction current $\gg$ displacement current
- poor conductor $\Rightarrow$ conduction current $\ll$ displacement current

In these lectures we will focus on good conductors for simplicit . In this case easiest to directly simplify Maxwell's Eqns to

$$① \quad \vec{\nabla} \cdot \vec{E} = 0$$

$$② \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$③ \quad \nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

$$④ \quad \nabla \times \vec{B} = \mu_r \mu_0 \sigma \vec{E}$$

"<u>Wave Equation</u>" <u>in Conductors</u>

Take $\vec{\nabla} \times$ (eqn ③) to give

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t} (\nabla \times \vec{B})$$

use ④

$$\vec{\nabla}(\nabla \cdot E) - \nabla^2 \vec{E} \qquad - \mu_r \mu_0 \sigma \frac{\partial E}{\partial t}$$

$\nabla \cdot E = 0$ by ①

Thus in a good conductor the analog of wave eqn is

$$\nabla^2 \vec{E} = + \mu_r \mu_0 \sigma \frac{\partial \vec{E}}{\partial t}$$

NOTE: NOT $\partial^2 / \partial t^2$

This is <u>NOT</u> a wave eqn!