

D Polarizable Materials

1. polarization: definition and physical origin

When an insulator (dielectric) is put in an electric field \underline{E} the field induces a dipole moment. This is measured by the polarization \underline{P} , defined as the dipole moment per unit volume defined over some averaging length scale d

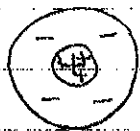
$$(\text{\AA} \ll d \ll L = \text{sample size})$$

An insulator (dielectric) has electrons 'held down' to their nuclei - cf. conductors where electrons are free to move through the material.

Why does the field induce a dipole moment?

(i) neutral atoms

electrons are displaced relative to the nucleus



no field



with field

field pulls +ve charges \rightarrow

Coulomb force pulls +ve charges \leftarrow

(field $\sim 10^4 \text{ V m}^{-1}$; displacement $\sim 10^{-18} \text{ m}$)
 \therefore tiny effect

as long as \underline{E} not too big, induced dipole moment $\propto \underline{E}$

$$\underline{p} = \alpha \underline{E}$$

\uparrow
 atomic polarizability

(ii) polar molecules

already have a dipole moment

in a field the moments will tend to point along the field

competing thermal effects will randomise the directions

small excess pointing along field and a polarisation $\propto \underline{E}$

(cf. Curie's law for paramagnets)

2. bound charge

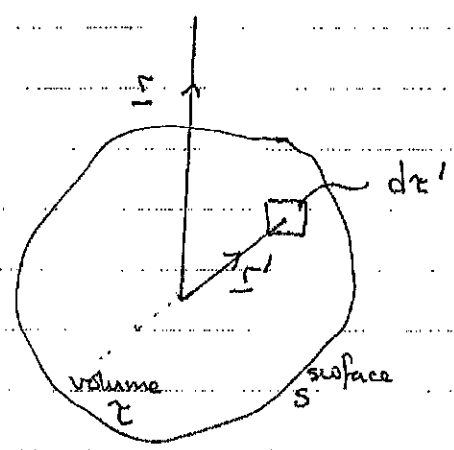
The potential (and \therefore field) of an object with polarization \underline{P} is the same as the potential produced by a volume charge density $\rho_b = -\text{div } \underline{P}$ plus a surface charge density $\sigma_b = \underline{P} \cdot \hat{n}$

\uparrow
unit normal to the surface

proof

potential at \underline{r} due to a dipole \underline{p} at \underline{r}'

$$V(\underline{r}) = \frac{\underline{p} \cdot (\underline{r} - \underline{r}')}{4\pi\epsilon_0 |\underline{r} - \underline{r}'|^2}$$



$$\underline{p} = \underline{P}(\underline{r}') d\tau'$$

\uparrow dipole moment per unit volume

\therefore potential at \underline{r} due to polarised object occupying τ is

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\underline{P}(\underline{r}') \cdot (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^2} d\tau'$$

using vector calculus
'integration by parts'

$$= \frac{1}{4\pi\epsilon_0} \int_{\tau} \underline{P}(\underline{r}') \cdot \text{grad}' \frac{1}{|\underline{r} - \underline{r}'|} d\tau'$$

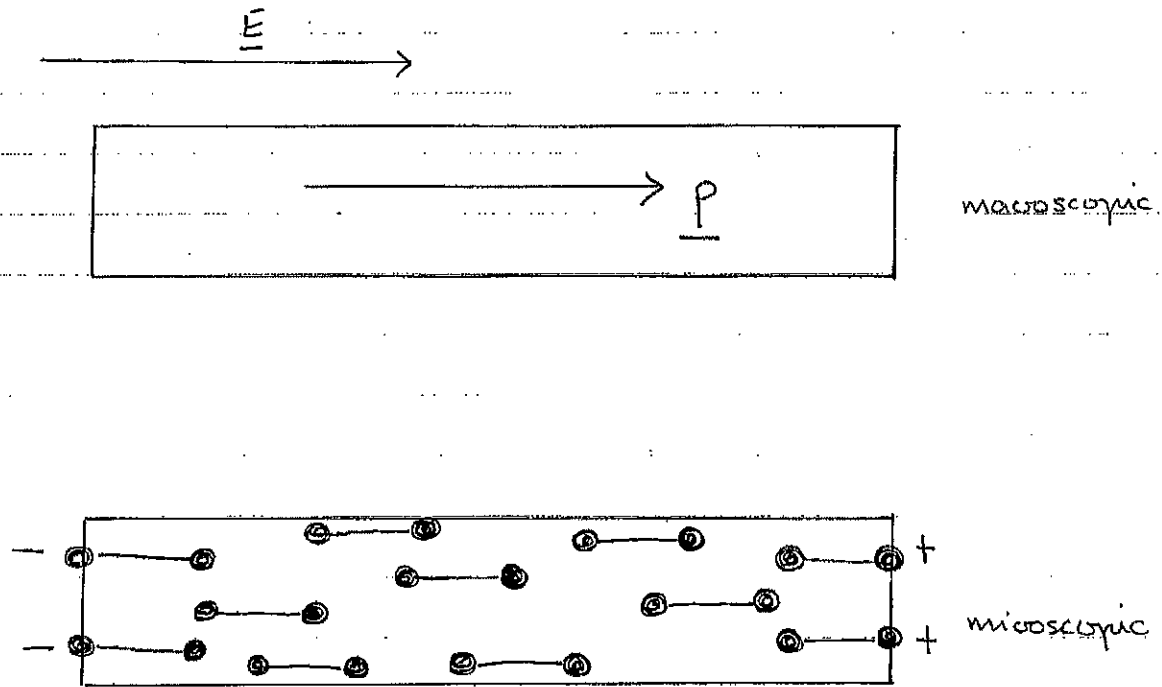
Gauss' Divergence Thm

$$= \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\text{div}' \{ \underline{P}(\underline{r}') \}}{|\underline{r} - \underline{r}'|} d\tau' - \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\text{div}' \underline{P}(\underline{r}')}{|\underline{r} - \underline{r}'|} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_S \frac{\underline{P}(\underline{r}') \cdot \hat{n}}{|\underline{r} - \underline{r}'|} dS' - \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\text{div}' \underline{P}(\underline{r}')}{|\underline{r} - \underline{r}'|} d\tau'$$

\uparrow
potential of a surface charge density $\sigma_b = \underline{P} \cdot \hat{n}$

\uparrow
potential of a volume charge density $\rho_b = -\text{div } \underline{P}$



$\rho_b = -\text{div } \underline{P}$: only get a contribution if \underline{P} varies with position otherwise +ve, -ve ends of dipoles cancel.

$\sigma_b = \underline{P} \cdot \hat{n}$: charge builds up on the surfaces.

\vec{P} is only well-defined on length scales much larger than the relevant microscopic length scale (eg, atomic sizes for solids, or inter-molecular distance for gases) as to get a value of $\vec{P}(\vec{r})$ that does not fluctuate wildly as we move around in the sample we need to average over all the atomic/molecular polarizations in a volume $\gg (\text{microscopic length})^3$

To be useful for experimental/practical considerations the averaging length scale must be sufficiently smaller than the overall sample size, ie $d^3 \ll L^3$

3. Gauss' law in dielectrics and the definition of \underline{D}

inside a dielectric

$$\text{div } \underline{E} = \frac{\rho_f + \rho_b}{\epsilon_0}$$

$$= \frac{\rho_f - \text{div } \underline{P}}{\epsilon_0}$$

$$\therefore \text{div } (\epsilon_0 \underline{E} + \underline{P}) = \rho_f$$

define $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$

$$\text{div } \underline{D} = \rho_f$$

NB. integral form is

$$\oint \underline{D} \cdot d\underline{s} = q_f$$

useful because formulae for \underline{D} involve only free charge
formula for \underline{E} must account for all (free and bound)
charge

Comment: Since \underline{P} is a macroscopic field defined by averaging so is \underline{D} - only in this way do we get a smooth field.

① electric displacement, usually just called ' \underline{D} '

4. Linear dielectrics and the relative permittivity, ϵ 

ie. $\underline{P} \propto \underline{E}$ ($\therefore \underline{D} \propto \underline{E}$)

write $\underline{P} = \epsilon_0 \chi_e \underline{E}$

↑ electric susceptibility

using ① $\therefore \underline{D} = \epsilon_0 (1 + \chi_e) \underline{E} \equiv \epsilon_0 \epsilon_r \underline{E}$ ②

↑ relative permittivity

from ① and ②

$$\underline{P} = (\epsilon_r - 1) \epsilon_0 \underline{E}$$
 ③

(N.B. Griffiths uses ' ϵ ' where I use ' $\epsilon_r \epsilon_0$ '
↑ sometimes)

5. Field due to a point charge in a linear dielectric

in dielectric
• rel. permittivity
 ϵ_r
 q_f

there will be bound charges, but writing Gauss' law for \underline{D} we can automatically include them.

$$\int_s \underline{D} \cdot d\underline{S} = q_f$$

↑
sphere

$$\therefore \underline{D} = \frac{q_f}{4\pi r^2} \hat{r}$$

$$\underline{E} = \frac{\underline{D}}{\epsilon_r \epsilon_0} = \frac{q_f}{4\pi \epsilon_r \epsilon_0 r^2} \hat{r}$$

\therefore the field and potential of a charge distribution in a linear dielectric is related to that in free space by writing $\epsilon_r \epsilon_0$ instead of ϵ_0 in the relevant formulas.

N.B. $\text{div } \underline{E} = \frac{\rho_f + \rho_b}{\epsilon_0}$ } always

$\text{div } \underline{D} = \rho_f$

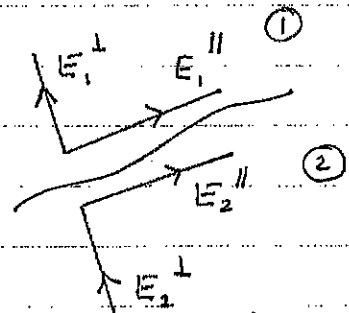
but $\underline{D} = \epsilon_r \epsilon_0 \underline{E}$ } linear dielectric

$\therefore \text{div } \underline{E} = \frac{\rho_f}{\epsilon_r \epsilon_0}$

Comment: The relative permittivity ϵ_r varies widely between substances. For low-density gases it $\epsilon_r \rightarrow 1$ (but be careful that avg. size still \gg intermolecular separation - otherwise whole concept of ϵ_r not well defined!). For solids/liquids typical values are $\epsilon_r \sim$ few but water has $\epsilon_r \approx 80$ and 'designed' materials much higher still (see Griffiths p. 180)

6. Boundary conditions at ~~and electric~~ boundary between dielectrics

from § 1A9



(i) $E_{1}^{\parallel} = E_{2}^{\parallel}$

followed from $\int \underline{E} \cdot d\underline{l} = 0$; always true in (magnet)static case / electro

(ii) $E_{1}^{\perp} - E_{2}^{\perp} = \frac{\sigma}{\epsilon_0}$

as $\partial \vec{B} / \partial t = 0$ so Faraday $\Rightarrow \nabla \times \vec{E} = 0$

followed from Gauss' thm; always true but need to remember that $\sigma = \sigma_f + \sigma_b$

so at a boundary between dielectrics usually easier to work with D. Gauss' thm for D gives

$D_{1}^{\perp} - D_{2}^{\perp} = \sigma_f$

but, for dielectrics, ^{in general,} $\sigma_f = 0 \therefore \underline{D}_{1}^{\perp} = \underline{D}_{2}^{\perp}$

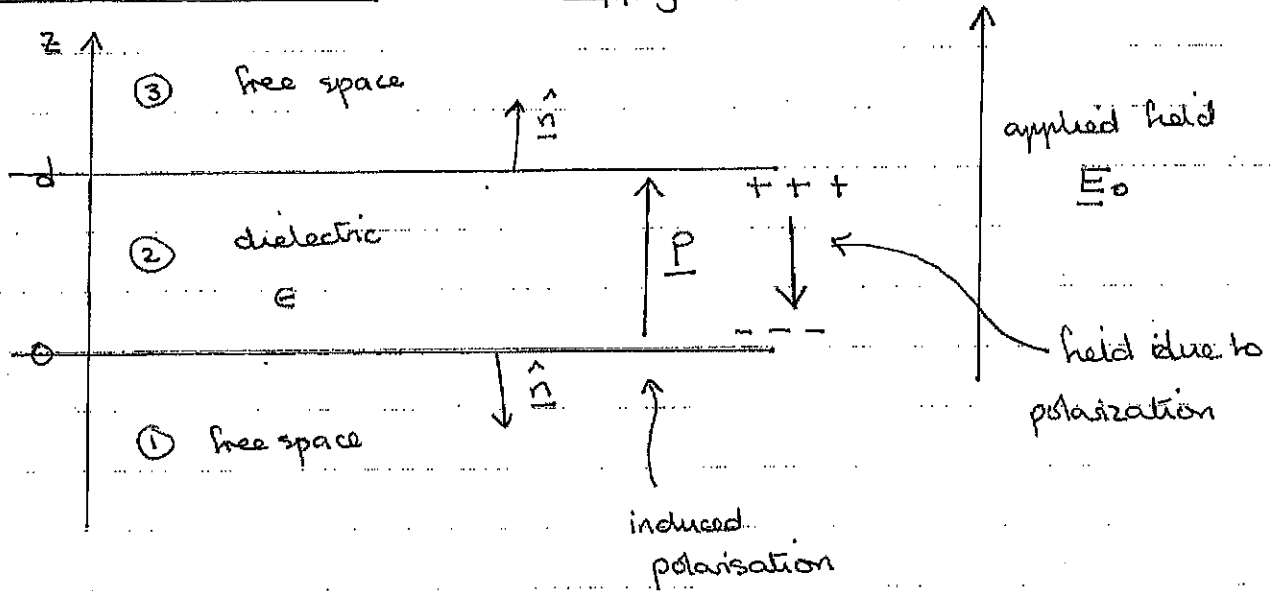
At a boundary between dielectrics the convenient boundary conditions are

E^{\parallel}	('E tangential')	continuous
D^{\perp}	('D normal')	continuous

Comment: $\nabla \times \vec{E} = 0 \not\Rightarrow \nabla \times \vec{D} = 0$ as many situations (even when all sources and fields are static) have $\nabla \times \vec{P} \neq 0$. This means that the parallel between $\nabla \cdot \vec{E} = \rho_{TOT} / \epsilon_0$ and $\nabla \cdot \vec{D} = \rho_f$ can be misleading (see p.178 and 182 of Griffiths)

linear dielectric slab

apply field \underline{E}_0



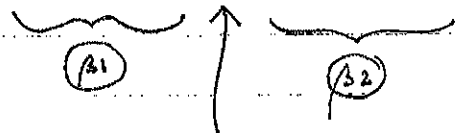
what are $\underline{D}_1, \underline{D}_2, \underline{D}_3$
 $\underline{E}_1, \underline{E}_2, \underline{E}_3$
 and in ② $\underline{P}, \rho_b, \sigma_b$?

①, ③ free space $\therefore \underline{E}_1 = \underline{E}_0, \underline{E}_3 = \underline{E}_0$

$\underline{D}_1 = \epsilon_0 \underline{E}_0, \underline{D}_3 = \epsilon_0 \underline{E}_0$

\underline{D}^\perp continuous $\therefore \underline{D}_2 = \epsilon_0 \underline{E}_0$ ②

but $\underline{D}_2 = \epsilon_r \epsilon_0 \underline{E}_2 = \epsilon_0 \underline{E}_2 + \underline{P}$



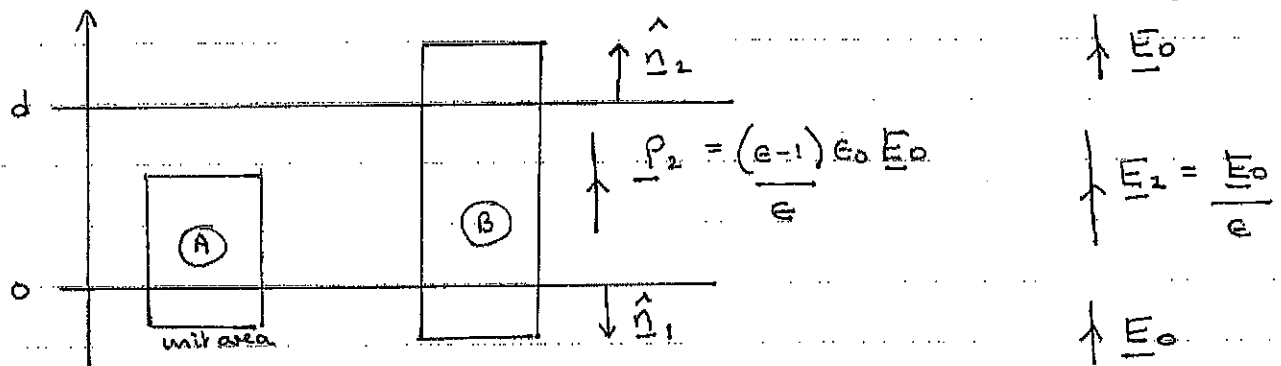
field actually in the dielectric $\neq \underline{E}_0$

from ② and (β1) $\underline{E}_2 = \frac{\underline{E}_0}{\epsilon_r} = \underline{E}_0 + \frac{(1-\epsilon_r)}{\epsilon_r} \underline{E}_0$

↑ external field
 ↑ field due to bound charge

from (β2) $\underline{P} = (\epsilon_r - 1) \epsilon_0 \underline{E}_2 = \frac{(\epsilon_r - 1)}{\epsilon_r} \epsilon_0 \underline{E}_0$

Does this all fit with what we know about bound charge?



\underline{P} constant $\therefore \rho_b = 0$

$$\text{at } z=0 \quad \sigma_b(0) = \underline{P}_2 \cdot \hat{n}_1 = -\frac{(\epsilon-1)\epsilon_0 \underline{E}_0}{\epsilon}$$

$$\text{at } z=d \quad \sigma_b(d) = \underline{P}_2 \cdot \hat{n}_2 = \frac{(\epsilon-1)\epsilon_0 \underline{E}_0}{\epsilon}$$

Gaussian surface (A)

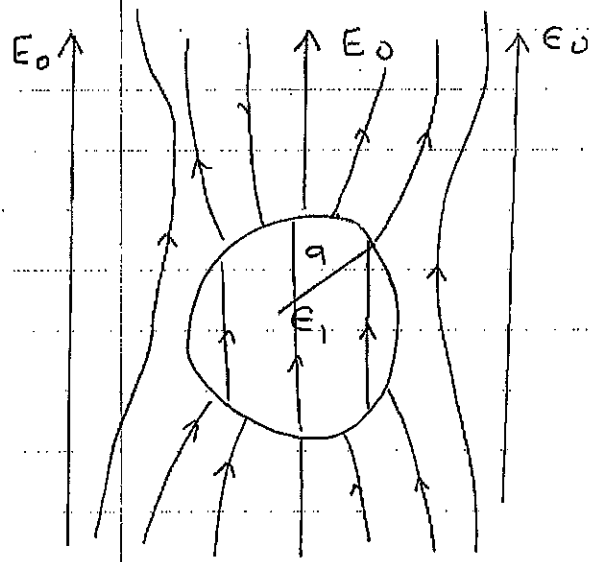
$$\int \underline{E} \cdot d\underline{S} = \int \frac{\rho}{\epsilon_0} dV$$

$$\frac{\underline{E}_0}{\epsilon} - \underline{E}_0 = -\frac{(\epsilon-1)\epsilon_0 \underline{E}_0}{\epsilon} \quad \checkmark$$

Gaussian surface (B)

$$0 = -\frac{(\epsilon-1)\epsilon_0 \underline{E}_0}{\epsilon} + \frac{(\epsilon-1)\epsilon_0 \underline{E}_0}{\epsilon} = 0 \quad \checkmark$$

7. Laplace revisited: dielectric sphere, relative permittivity ϵ_1 , radius a in a uniform field. Find V everywhere.



Laplace, spherical coords, azimuthal symmetry

⇓

$$V(r, \theta) = \sum_l \left\{ A_l r^l + \frac{B_l}{r^{l+1}} \right\} P_l(\cos \theta)$$

boundary conditions

- V_{in} finite at origin. ①
- $V_{out} \rightarrow -E_0 r \cos \theta$ as $r \rightarrow \infty$ ②
- at $r = a$ E_{\parallel} continuous ③
- D_{\perp} continuous ④

need separate solutions for V_{in} , V_{out}
need to match ' $\cos \theta$ ' term \therefore guess $l=1$ terms needed.

$$V_{in} = A_1 r \cos \theta \quad \checkmark \text{ ① no } \frac{1}{r^2} \text{ term}$$

$$V_{out} = -E_0 r \cos \theta + \frac{B_1}{r^2} \cos \theta \quad \checkmark \text{ ②}$$

$$\underline{E} = -\text{grad } V = \left(-\frac{\partial V}{\partial r}, -\frac{1}{r} \frac{\partial V}{\partial \theta}, -\frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \right)$$

$$\therefore \underline{E}_{in} = \left(-A_1 \cos \theta, A_1 \sin \theta, 0 \right)$$

$\uparrow E_{\perp}$ $\uparrow E_{\parallel}$

$$\underline{E}_{out} = \left(E_0 \cos \theta + \frac{2B_1}{r^3} \cos \theta, -E_0 \sin \theta + \frac{B_1}{r^3} \sin \theta, 0 \right)$$

$\underline{D}_{in} = \epsilon \epsilon_0 \underline{E}_{in}$ $\underline{D}_{out} = \epsilon_0 \underline{E}_{out}$

③ $\Rightarrow A_1 = -E_0 a + \frac{B_1}{a^3}$

④ $\Rightarrow -\epsilon \epsilon_0 A_1 = \epsilon_0 E_0 + \frac{2\epsilon_0 B_1}{a^3}$

solve for A_1, B_1 and sub into expressions for V :

⑤ $V_{in} = \frac{-3\epsilon_0}{\epsilon_1 + 2} r \cos \theta = -E_0 r \cos \theta + E_0 r \cos \theta \left(\frac{\epsilon_1 - 1}{\epsilon_1 + 2} \right)$

⑥ $V_{out} = -E_0 r \cos \theta + \frac{\epsilon_0 a^3}{r^2} \left(\frac{\epsilon_1 - 1}{\epsilon_1 + 2} \right) \cos \theta$

contribution due to external field

contribution due to polarization of sphere
dipolar outside
constant field inside

Two questions

(i) what is \underline{P} in the sphere?

$\underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon \epsilon_0 \underline{E}$

$\therefore \underline{P} = (\epsilon_1 - 1) \epsilon_0 \underline{E}_{in} = \frac{(\epsilon_1 - 1) \epsilon_0 \cdot 3 E_0}{\epsilon_1 + 2} \hat{z}$

(ii) What is the bound charge on the surface?

$\sigma_b = \underline{P} \cdot \hat{n} = P \cos \theta = \frac{3 \epsilon_0 (\epsilon_1 - 1) E_0 \cos \theta}{\epsilon_1 + 2}$