Non-Abelian gauge theories

// We start with vector multiplet V. The difference:

V has components for each generator: $V = V_a T^a_{ij}$

Chiral superfields: $\Phi \to e^{-i\Lambda}\Phi$

$\phi^+ \to \phi^+ e^{i\Lambda^+}$

$\Lambda = T^a_{ij}\Lambda_a$

generator in the representation of ϕ

So:

$$\Phi^+ e^{2V}\Phi \to \phi^+ e^{i\Lambda^+} e^{2V'} e^{-i\Lambda}\Phi$$

If $e^{2V'} = e^{i\Lambda^+} e^{2V} e^{i\Lambda}$, the interaction is invariant.

$V' = V + \frac{i}{2}(\Lambda - \Lambda^+) + \ldots$

commutators from evaluating Baker-Campbell-Hausdorff

after evaluating many commutators $[T^a, T^b] = f^{abc}T^c$

one can get reduced expressions $V' = V'_a T^a_{ij}$

* With such an "explicit form" the transformation of V is independent of the specific matter representation.
* We still can go to Wess-Zumino gauge

$$\int d^4\theta\, \phi^\dagger e^{2V}\phi$$

Field strength superfield:

$$W_\alpha := -\frac{1}{8}\bar{D}\bar{D}\, e^{-2V} D_\alpha e^{2V}$$

Transformation behaviour:

$$e^{-2V} e^{2V} = 1 \quad \& \quad e^{2V'} = e^{-i\Lambda^\dagger} e^{2V} e^{i\Lambda}$$

$$\Rightarrow e^{-2V'} = e^{-i\Lambda} e^{-2V} e^{i\Lambda^\dagger}$$

$$W_\alpha \to \underset{\substack{\bar{D}_{\dot\alpha}\phi=0 \\ D_\alpha\phi=0}}{\frac{1}{8}} \bar{D}\bar{D}\left(e^{-i\Lambda} e^{-2V} e^{i\Lambda^\dagger}\right) D_\alpha\left(e^{-i\Lambda^\dagger} e^{2V} e^{i\Lambda}\right)$$

$$= e^{-i\Lambda} \frac{1}{8} \bar{D}\bar{D}\left(e^{-2V}\right)\left[\left(D_\alpha e^{2V}\right) e^{i\Lambda} + e^{2V} D_\alpha e^{i\Lambda}\right]$$

$$= e^{-i\Lambda} \; W_\alpha \; e^{i\Lambda}$$

$\Rightarrow$ $\mathrm{Tr}(W^\alpha W_\alpha)$ is gauge invariant and the Lagrangian is:

$$\int d^2\theta \; f(\phi) \, \mathrm{Tr}(W^\alpha W_\alpha) + \text{h.c.}$$

Let's look at further modifications:

× $F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu + [A'_\mu, A_\nu]$

× non-Abelian covariant derivatives $D_\mu \phi = \partial_\mu \phi + A_\mu^a T^a \phi$

× D-term: $\phi^\dagger e^{2V} \phi \to D^a \phi_i^\dagger T^a \phi_i$

$$\Rightarrow \frac{(D^a)^2}{2g^2} + D^a \phi_i^\dagger T^a \phi_i$$

$$\Rightarrow V_{D_a} = \frac{g^2}{2} \sum_i \left(\phi_i^\dagger T_a \phi_i\right)^2$$

No FI-term

Non-renormalisation theorem

in generic QFT, quantum corrections to classical (tree-level) action renormalise couplings in Lagrangian order by order in perturbation theory (+ non-perturbative corrections).

effective Wilsonian action below cut-off Λ:

$$e^{iS_{eff}} = \int_{p>\Lambda} D\phi \, e^{iS(\phi)}$$

effective Lagrangian has masses, couplings depending on Λ (organised by an expansion in loops of Feynman diagrams)

usually infinite series of terms (all terms allowed by symmetries of the theory)

eff. lagrangian reproduces same results as original L for processes at momenta below Λ

How do SUSY theories (K, W, f, ξ) behave under quantum corrections?

We will show:

- K gets corrections order by order in perturbation theory
- only one-loop corrections for $f(\phi)$
- W & ξ are not renormalised in perturbation theory

History: 1977 Grisaru, Rocek, Siegel using "supergraphs"

1993 Seiberg (based on string theory arguments by Witte 1985) using symmetry + holomorphicity arguments

Proof:

Promote Coupling constants to fields (naturally appears in string theory, which are very weakly coupled

Ansatz:

$$\mathcal{L} = \int d^4\theta \left(K + \xi V_{U(1)}\right) + \int d^2\theta \; Y W(\phi_i) + X W^\alpha W_\alpha + h.c.$$

X, Y chiral superfields

$\mathcal{L}$ reduces to standard Lagrangian if scalar components $x = f(\phi)$ $y = 1$ & remaining components vanish

(Assume) SUSY + gauge invariance preserved, most general Lagrangian

$$\mathcal{L}_X = \int d^4\theta \; J(\phi, \phi^\dagger, e^V, X, Y, \mathcal{D}, \dots) + \xi(X, X^\dagger, Y, Y^\dagger) V_{U(1)}$$

derivatives

$$+ \int d^2\theta \; \underbrace{H(\phi, X, Y, W^\alpha)}_{\text{holomorphic}} + h.c.$$

Symmetries:

- J, H gauge invariant
- R-symmetry: fields and θ-coordinates have different $U(1)_R$ charges

fields	θ	$\bar\theta$	$\int d\theta$	ϕ	V	Y	$W = \bar{\mathcal{D}}\bar{\mathcal{D}} \cdot \mathcal{D}$	X
R-charge	1	-1	-1	0	0	2	1	0

$\int d\theta\, \theta = 1$ $\quad \int d^2\theta\, W \to$

- Peccei-Quinn symmetry $\quad X \to X + ir \quad r \in \mathbb{R}$

$$X W^\alpha W_\alpha \supset \mathrm{Re}\{X\} F_{\mu\nu} F^{\mu\nu} + \underbrace{\mathrm{Im}\{X\} F_{\mu\nu} \tilde{F}^{\mu\nu}}_{\text{topological term}}$$

Change in imaginary of X would only add total derivative locally to $\mathcal{L}$ + vanishes for field configs. continuously conn. to vacuum

$$\mathcal{L} \to \mathcal{L} + r F_{\mu\nu} F^{\mu\nu}$$

*

$U(1)_R$: $\quad H = Y\, h(X,\phi) + g(X,\phi)\, W^\alpha W_\alpha \qquad (*)$

Shifts: $\quad h = h(\phi)$

$$g(X,\phi) = \left(\alpha \underset{\substack{\uparrow \\ \text{linear}}}{X} + g(\phi)\right) W^\alpha W_\alpha$$

Limits: Reason for introducing X & Y is that by taking appropriate limits we can use our knowledge @ weak coupling to determine couplings in $(*)$.

$Y \to 0$ $\quad$ all couplings in superpotential vanish as Y

$Yh = YW$ $\quad$ so $h(\phi) = W(\phi)$

$\Rightarrow$ W not renormalised

$\star$ So when we perform path integral

$$\int \mathcal{D}A\, e^{\frac{1}{g^2} F_{\mu\nu}F^{\mu\nu} + i\theta F_{\mu\nu}\star F^{\mu\nu}}$$

Non-perturbat instantonic terms do depend on θ and behave like $e^{-\frac{8\pi^2}{g_{YM}^2} + i\theta}$

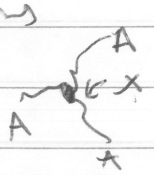
The perturbative expansion depends on $\frac{1}{g^2} = \phi + \bar{\phi}$ and not on θ

$$\left(\alpha x + g(\phi)\right) W^\alpha W_\alpha$$

$x \to \infty$: let's count powers of x:

* gauge propagators: $\sim \frac{1}{x}$ [inverse of kinetic term $\times \partial_\mu A_\nu \partial^\mu A^\nu$]

* gauge interactions



Number of powers of x:

$$N_x = \#V_w - I_w$$

$\#V_w$: #vertices; I_w: internal lines

Number of loops L:

$$L = I_w - V_w + 1$$

$$\Rightarrow N_x = 1 - L$$

$L = 0$ (tree-level) $\Rightarrow N_x = 1$ ($\alpha = 1$) to reproduce tree-level result

$L = 1$ (one-loop) $\Rightarrow N_x = 0$

$\Rightarrow$ Loop-corrected gauge coupling:

$$x + g(\phi)$$

x: tree; $g(\phi)$: 1-loop

- Kähler potential: $\mathcal{J}(Y Y^\dagger, X + X^\dagger, \dots)$
  no restrictions

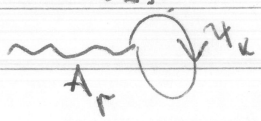
- $\overline{+}\xi$-term:

$$\xi(X, X^\dagger, Y, Y^\dagger)\, V_D^{U(1)}$$

gauge invariance: $V \to i(\Lambda - \Lambda^\dagger) + V$

$\Rightarrow \xi$ constant (independent of $X, X^\dagger, Y, Y^\dagger$)

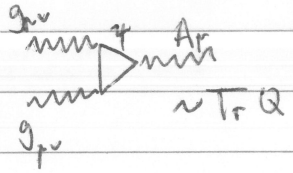
Only contributions:



(tadpole diagrams)

$$\sim \sum q_i = \mathrm{Tr}\, Q_{U(1)}$$

if $\neq 0$ so-called gravitational anomaly present (inconsistent).



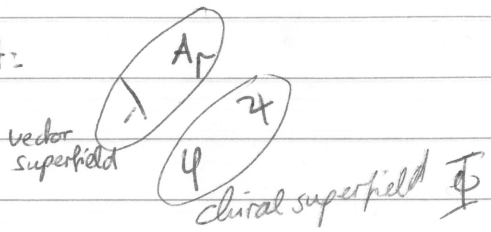
(if they were present: gauge invariance would be broken

$\Rightarrow$ If no gravitational anomaly, FI-term not-renormalized

SUSY Lagrangians for extended SUSY

$N=1$: special $N=0$ Lagrangian
$N=2$: special $N=1$ Lagrangian

vector-multiplet:



⇒ $W=0$ in $N=2$

K & f can be written in terms of a single hol. function: prepotential $\mathcal{F}$

$$f(\Phi) = \frac{\partial^2 \mathcal{F}}{\partial \Phi^2} \qquad K(\Phi, \Phi^\dagger) = \frac{1}{2i}\left(\Phi^\dagger \exp(2V) \frac{\partial \mathcal{F}}{\partial \Phi} - \text{h.c.}\right)$$

$\mathcal{F}$ ... only 1-loop pert. corrections
but there is more $\mathcal{F}_{\text{non-pert.}}$ (e.g. Seiberg, Witten)
+ hyper-multiplets

$N=4$

$$\begin{pmatrix} \lambda & A_\mu & \psi_1 \\ & \psi_1 & \end{pmatrix} + \begin{pmatrix} \psi_3 & \psi_2 & \phi_2 \\ & \psi_3 & \end{pmatrix}$$

$N=2$ vector $\qquad$ $N=2$ hyper

no free functions, only 1-parameter

$$f = \tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$$

$N=4$ finite theory, $\beta = 0$
couplings remain constant at any scale ⇒ conformal invariance
S-duality: $\tau \to \frac{a\tau + b}{c\tau + d}$ $\qquad (SL(2,\mathbb{Z}))$

SUSY breaking

Gauge theory breaking: $\varphi_i \mapsto \exp(i\alpha^a T^a)_{ij}\varphi_j$

$\delta\varphi_i = i\alpha^a T^a \varphi$

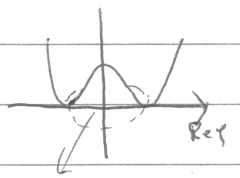
gauge symmetry broken if vacuum state φ_{vac} transforms in non-trivial way:

$$(\alpha^a T^a)_{ij}(\varphi_{vac})_j \neq 0$$

$U(1)$: $\varphi = \rho e^{i\nu}$

$\delta\varphi = i\alpha\varphi$

$\Rightarrow \delta\rho = 0$ & $\delta\nu = \alpha$



$\delta\nu = \alpha$ ⇑ Goldstone boson

~~Broken~~ SUSY defined as: $Q_\alpha|vac\rangle \neq 0$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu$$

$$\Rightarrow (\bar\sigma^\nu)^{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\bar\sigma^\nu)(\sigma^\mu) P_\mu = 4\eta^{\mu\nu} P_\mu = 4P^\nu$$

$\nu=0$ component: $(\sigma^0)^{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \sum_{\alpha=1}^{2}\left(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha\right) = 4P^0 = 4E$

($\sigma^0 = \mathbb{1}$)

Implications: ~~$\langle\varphi|Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha|\varphi\rangle \neq$~~

- $Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha$ positive semi-definite $\Rightarrow E \geq 0$ for any state.
- $\langle vac|Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha|vac\rangle > 0$, so if broken SUSY, the energy is strictly positive, $E > 0$.

Equivalent of $\langle\varphi\rangle \neq 0$, $E > 0$ in SUSY vacuum.

↓ gauge symm. breaking

F-term breaking

Transformation of chiral superfield ϕ:

$$\delta\varphi = \sqrt{2}\,\xi\psi$$
$$\delta\psi = \sqrt{2}\,\xi F + i\sqrt{2}\,\sigma^\mu \bar{\xi}\,\partial_\mu \varphi$$
$$\delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu \partial_\mu \psi$$

What's the equivalent of $\delta\psi \neq 0$?

If $\delta\varphi, \delta\psi$, or $\delta F \neq 0$ susy is broken.

We are interested in Lorentz-invariant theories:

$$\langle \psi \rangle = \langle \partial_\mu \varphi \rangle = 0$$

Why? Non-trivial rep under Lorentz group

So:

$$\text{susy} \iff \langle F \rangle \neq 0$$

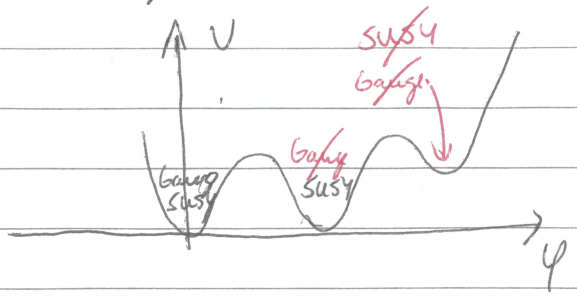
↪ only fermionic part of ϕ will change:

$$\delta\psi = \sqrt{2}\,\xi F \neq 0$$

ψ is called a Goldstone fermion or the goldstino (not partner of some Goldstone boson).

$$V_F = K^{-1}_{ij} \frac{\partial W}{\partial \varphi_i} \frac{\partial W^*}{\partial \varphi_j^*} = K^{-1}_{ij} F_i \overline{F_j}$$

↪ ~~Susy~~ $\iff$ $V_F > 0$



O'Raifertaigh model

3 chiral superfields ϕ_1, ϕ_2, ϕ_3

$$K = \phi_i^\dagger \phi_i$$
$$W = g\phi_1(\phi_3^2 - m^2) + M\phi_2\phi_3 \qquad M \gg m$$

F-term eqns:

$$-F_1^* = \frac{\partial W}{\partial \phi_1} = g(\phi_3^2 - m^2)$$
$$-F_2^* = \frac{\partial W}{\partial \phi_2} = M\phi_3$$

} cannot vanish simultaneously $\exists F \neq 0$

$$-F_3^* = \frac{\partial W}{\partial \phi_3} = 2g\phi_1\phi_3 + M\phi_2$$

This model breaks SUSY.

Let's look at spectrum:

$$V = \left(\frac{\partial W}{\partial \phi_i}\right)\left(\frac{\partial W}{\partial \phi_j}\right)^* \delta_{ij} = g^2|\phi_3^2 - m^2|^2 + M^2|\phi_3|^2 + |2g\phi_1\phi_3 + M\phi_2|^2$$

If $m^2 < \frac{M^2}{2g^2}$; minima at:

$\langle\phi_2\rangle = \langle\phi_3\rangle = 0$, $\langle\phi_1\rangle$ arbitrary

← such a field is called a modulus, $\Rightarrow m_{\phi_1} = 0$

For simplicity take $\langle\phi_1\rangle = 0$ & compute spectrum of fermions and scalars.

Fermion mass term: $\left(\frac{\partial^2 W}{\partial\phi_i \partial\phi_j}\right)\psi_i\psi_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix}\psi_i\psi_j$

$\Rightarrow$ $m_{\psi_1} = 0$, $m_{\psi_2} = m_{\psi_3} = M$ (appropriate rotation of ψ_2, ψ_3)

ψ_1 Goldstino as $\delta\psi_1 \sim \langle F_1\rangle \neq 0$