QFT in Curved Spacetimes, TT 2017. Problem Set 1.
Due date: May 25 at 12pm

1. Adapt the derivation of the relation between spin and statistics done in the lecture to the case of Fermions. Once again, you can assume a space-time of cosmological form with two regions in which the scale factor becomes constant. Note that for Fermions the relevant inner product does not involve derivatives since the Lagrangian is of first order.

2. For the cosmological example studied in class, show by explicit construction that there is a unitary operator which relates the vacua $|0\rangle_{in}$ and $|0\rangle_{out}$. *Hint:* the operator has the schematic form $S \sim \exp(a^\dagger a^\dagger + aa)$.

3. Consider a minimally coupled massive scalar field in a 2 dimensional spacetime with metric given by

$$ds^2 = dt^2 - a^2(t)dx^2 \tag{1}$$

Defining the conformal time η via $dt = ad\eta$, take the scale factor to be of the form

$$a^2(\eta) = \kappa_1 + \kappa_2 \tanh(\eta/\rho) \tag{2}$$

Note that this defines two asymptotics regions $\eta \to \pm\infty$ in which the spacetime becomes Minkowski space. Denoting the corresponding regions by 'in' and 'out' as usual, find the explicit form of the Bogolubov transformation that relates the corresponding creation and annihilation operators. *Hint:* the wave equation can be solved explicitly in terms of Hypergeometric functions.

4. Consider a minimally coupled massive scalar ϕ with mass squared m^2 field in 4-dimensional Anti-de Sitter (AdS) spacetime, the line element of which is given by

$$ds^2 = (1+r^2)dt^2 - \frac{dr^2}{(1+r^2)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3}$$

We are using units in which the AdS length is set to 1. The boundary of the spacetime corresponds to the region $r \to \infty$. Consider a mode expansion of the field of the form

$$\phi(x^\mu) = e^{-i\omega t} Y_{\ell m}(\theta, \phi) R(r) \tag{4}$$

a) Show that for large r, the profiles have the form $R \sim r^{-\Delta_\pm}$. Reality of $\Delta_\pm$ puts a lower bound on the fields' mass, determine this value.
b) As opposed to flat spacetimes, AdS is non-globally hyperbolic, which in practice means that you have to specify boundary conditions at infinity in order to determine the classical dynamics and quantize the theory. Show that for $m^2 > 0$, the only boundary conditions compatible with finiteness of the symplectic product are to set the slow fall off of the field to zero.
c) Imposing the boundary conditions of b) and regularity at the interior, find the spectrum of frequencies and the corresponding eigenmodes. *Hint:* once again, the general solution of the wave equation can be expressed in terms of Hypergeometrics. These simplify drastically when evaluated on the particular eigenfrequencies.
d) Show that modes associated to different frequencies are orthogonal with respect to the Klein-Gordon inner product. Explain how this is required by conservation of the symplectic product.