QFT in Curved Spacetimes, TT 2017. Problem Set 1.
Due date: May 11 at 12pm

1. When writing the mode expansion of a field in terms of ladder operators, we declare that $a_i^\dagger$ are the coefficients of the negative frequency solutions and a_i the positive frequency solutions, with the vacuum satisfying $a_i|0\rangle = 0$. Explain what happens if the opposite choice is made.

2. The Klein-Gordon norm is a particular case of a symplectic structure, which can be defined for any field theory possessing a Lagrangian formulation. The definition is as follows:

i) For a general variation $\delta_1\phi$, define the pre-symplectic potential $\theta^\mu(\delta_1)$ as the

$$\delta\mathcal{L} = (\text{eom})\delta_1\phi + \partial_\mu\theta^\mu \tag{1}$$

where (eom) denotes the equations of motion.

ii) Compute the anti-symmetrized variation of $\theta^\mu(\delta_1)$, termed the pre-symplectic current

$$\omega^\mu(\delta_1\phi, \delta_2\phi) := \delta_2\theta^\mu(\delta_1\phi) - \delta_1\theta^\mu(\delta_2\phi) \tag{2}$$

The integral of ω^μ over a space-like surface Σ with unit normal n_μ is then the pre-symplectic form

$$\Omega[\delta_1\phi, \delta_2\phi] = \int_\Sigma n_\mu\omega^\mu(\delta_1\phi, \delta_2\phi) \tag{3}$$

In a gauge theory, Ω has null directions corresponding to the gauge transformations. The symplectic structure is then obtained by defining Ω to act on the quotient space of the physical modes, i.e. all modes modulo gauge transformations. With this in mind, one usually omit the prefix (pre)-symplectic structure, understanding that gauge modes could be present. Just like the Klein-Gordon product, this construction provides a tool to quantize more general field theories.

a) Compute the symplectic current for the case of an Abelian $U(1)$ gauge field in flat space, with Lagrangian given by $\mathcal{L} = -1/4F_{\mu\nu}F^{\mu\nu}$.
b) Show that the symplectic current is closed on-shell, i.e. $\partial_\mu\omega^\mu = 0$.
c) Use b to show that Ω is conserved, i.e. independent of Σ, when the appropriate boundary conditions hold.
d) Show that gauge transformations $\delta_{gauge}A$ are null directions of Ω, i.e. that $\Omega[\delta_{gauge}A, \delta A] = 0$ for all solutions δA.

3. Consider the scalar field action in a general curved space-time of dimension D endowed with metric g

$$S = \frac{1}{2}\int d^Dx\sqrt{|g|}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2) \tag{4}$$

where R is the Ricci scalar of g. Consider a conformal transformation

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \qquad \tilde{\phi}(x) = \Omega^{2p}(x)\phi(x) \tag{5}$$

a) Find the values of the parameters for which the theory is conformal invariant in any D.

b) Compute the stress tensor of the theory, $T^{\mu\nu}$, defined as

$$\delta S = -\frac{1}{2}\int d^D x\sqrt{|g|}T^{\mu\nu}\delta g_{\mu\nu} \tag{6}$$

Show that it is conserved for all values of ξ and m^2, and that it is traceless for the special case determined in a).

4. Consider a Maxwell field in a general curved space-time of dimension D endowed with metric g

$$S = \frac{1}{4}\int d^D x\sqrt{|g|}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} \tag{7}$$

a) Derive the equations of motion and the stress tensor of the theory

b) Show by explicit calculation that the stress tensor is conserved on shell

c) In which dimension is the stress tensor traceless? why?