

Handout 12

Magnetoresistance in two-dimensional systems and the quantum Hall effect

12.1 Introduction: two dimensional systems

Lectures 9 and 11 described the effects of a magnetic field on three-dimensional carrier systems; in particular, Lecture 9 discussed Landau quantisation. We shall now look at two-dimensional semiconductor systems in an analogous manner. The two-dimensionality removes one degree of freedom, so that at high magnetic fields the Landau quantisation results in a density of states consisting entirely of discrete levels, rather than a series of one-dimensional density-of-state functions. This means that the oscillatory phenomena seen in high magnetic fields are often far more extreme in two-dimensional systems than they are in the three-dimensional equivalent.

The reduced dimensionality also results in some other novel phenomena; the quantum Hall effect is a general low-temperature property of two-dimensional or quasi-two-dimensional systems and has been observed in semiconductor structures (*e.g.* Si MOSFETs, GaAs-(Ga,Al)As heterojunctions and quantum wells containing two-dimensional holes or electrons) and, very recently, in some organic molecular conductors. Most of the research on the quantum Hall effect has been carried out using modulation-doped GaAs-(Ga,Al)As heterojunctions, as these can provide an exceptionally clean two-dimensional electron system with a very low scattering rate (see Lecture 8). We shall therefore confine our discussion to electrons in GaAs-(Ga,Al)As heterojunctions, although similar principles will apply to any of the other systems mentioned above. The effective mass of the electrons in the heterojunction will be labelled m^* , and their areal carrier density N_s .

In a heterojunction, the motion of the electrons in the growth (z) direction is confined by the approximately triangular potential well at the interface; in this direction, the electrons are in bound states of the well known as *subbands*. We let the j th subband have energy E_j . If a magnetic field is applied in the z direction, then the electrons' motion in the xy plane will also become quantised into Landau levels, with energy $(l + \frac{1}{2})\hbar\omega_c$. Therefore, the total energy of the electron becomes

$$E(B, l, j) = (l + \frac{1}{2})\hbar\omega_c + E_j, \quad (12.1)$$

i.e. the system has become completely quantised into a series of discrete levels.

(We remark at this stage that the magnetic field has reduced the effective dimensionality of the electron system by two; its density of states is now like that of *e.g.* an atom, with no electronic dispersion whatsoever. This is a general property of a magnetic field; in the case of a three-dimensional system, the electronic density of states becomes a series of density of states functions for one-dimensional motion.)

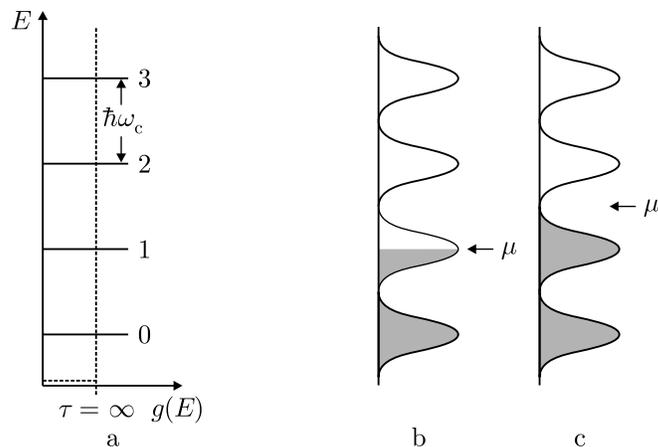


Figure 12.1: The Landau level density of states with zero scattering (left) and in the presence of broadening due to scattering (right). The shaded portions indicate states filled with electrons. Two positions of the chemical potential (Fermi energy) are shown.

12.2 Two-dimensional Landau-level density of states

Let us assume for the time being that only the lowest subband is populated with electrons. The density of states, which is a series of delta functions separated by $\hbar\omega_c$, is shown in Figure 12.1a. In a real sample, these levels will not be perfectly sharp, but instead broadened due to the scattering of electrons by imperfections at the surface of the layer and the presence of impurities. If the typical relaxation time between scattering events is τ (*i.e.* the lifetime of a particular quantum-mechanical state is τ), the Uncertainty Principle leads to broadening of the levels by an amount $\delta E = \hbar/\tau$. The effects of the magnetic field will therefore only be important if the energy separation of the Landau levels, $\hbar\omega_c$, is larger than this broadening (*i.e.* $\omega_c\tau \gg 1$). We shall concentrate on this high-field regime of well-separated Landau levels.

Since the number of electronic states at zero field has to equal the number at finite field, the number of electrons which it is possible to put into each Landau level may be calculated from the number of free electron wavevector states with energies between adjacent Landau levels. The wavevector k_l with zero field energy corresponding to the l th Landau level is given by

$$\frac{\hbar^2 k_l^2}{2m^*} = \left(l + \frac{1}{2}\right)\hbar\omega_c. \quad (12.2)$$

We divide the area of wavevector-space between k_l and k_{l+1} by the well-known degeneracy of two-dimensional phase-space ($N = 2(2\pi)^{-2}A_r A_k$, where A_r is the area in r -space, A_k is the area in k -space and the factor 2 takes into account spin) to get the number of states per unit area per Landau level

$$\frac{N}{A} = 2 \frac{\pi k_{n+1}^2 - \pi k_n^2}{(2\pi)^2} = \frac{2eB}{h}. \quad (12.3)$$

Each Landau level can hold at most this number of electrons before it is necessary to start to fill the next level.

The electrical properties of the sample will depend very strongly upon how the Landau levels are occupied. Let us assume first of all that the temperature is absolute zero, and that the highest occupied Landau level is half full (Figure 12.1b), *i.e.* the chemical potential μ is in the middle of a Landau level. Under such conditions, the system looks like a metal; there are lots of empty states which are just above μ , and so the system will have a high electrical conductivity.

Next, consider what happens when the highest filled Landau level is completely filled (Figure 12.1c). In this case, μ will be in the gap between the highest occupied and lowest unoccupied Landau level; there will be no empty states available at energies close to μ . This situation is analogous to that of an

insulator; the conductivity of the sample will obviously be a minimum under such circumstances, and for well-separated Landau levels will be effectively zero.

Finite temperatures will cause the Fermi–Dirac distribution function to become “smeared out” close to μ ; however, as long as $\hbar\omega_c \gg k_B T$, the general picture remains true that the conductivity oscillates dramatically as the field is swept. The conductivity will be very small whenever we have a completely filled highest Landau level, *i.e.* when

$$N_s = j \frac{2eB}{h}, \quad (12.4)$$

where j is an integer and N_s is the number of electrons per unit area. The maximum conductivity case occurs when

$$N_s = \left(j + \frac{1}{2}\right) \frac{2eB}{h}. \quad (12.5)$$

The conductivity will therefore oscillate as a function of magnetic field, and will be periodic in $1/B$, with a periodicity given by

$$\Delta(1/B) = \frac{2e}{hN_s}. \quad (12.6)$$

12.2.1 Resistivity and conductivity tensors for a two-dimensional system

The experimental geometry for the quantum Hall effect is usually exactly the same as that for the conventional Hall effect (see Figure 11.1). Thus, it is the components of the resistivity tensor that we measure, and not the conductivity. We can relate the current densities J_x and J_y (in two dimensions current densities will have the units *current per unit length*) and electric fields E_x and E_y by the expressions

$$J_x = \sigma_{xx}E_x + \sigma_{xy}E_y \quad (12.7)$$

and

$$J_y = -\sigma_{xy}E_x + \sigma_{xx}E_y \quad (12.8)$$

since $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$ for a homogeneous isotropic substance. If, as usual, we allow a current to flow along only one direction (x) (see Figure 11.1), then $J_y = 0$ and we can say that $E_y/E_x = \sigma_{xy}/\sigma_{xx}$. Therefore we can define the resistivities

$$\rho_{xx} \equiv \frac{E_x}{J_x} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \quad (12.9)$$

and

$$\rho_{xy} \equiv \frac{E_y}{J_x} = \frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}. \quad (12.10)$$

Note that the units of ρ_{xx} are known as Ω per square; a dimensionless geometrical factor (sample length/sample width) must be introduced to work out the resistance of a particular sample.

We use the Relaxation-Time Approximation to work out the motion of the electrons, *i.e.*

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{e}{m^*} \mathbf{E} - \frac{e}{m} \mathbf{v} \times \mathbf{B} - \frac{\mathbf{v}}{\tau}. \quad (12.11)$$

In steady state, the LHS vanishes, and the two components of the equation read

$$v_x = -\frac{e\tau}{m^*} E_x - \omega_c \tau v_y \quad (12.12)$$

and

$$v_y = -\frac{e\tau}{m^*} E_y + \omega_c \tau v_x, \quad (12.13)$$

where we have written $\omega_c = eB/m^*$. If we impose the condition $v_y = 0$ (no current in the y direction) we have

$$\frac{E_y}{E_x} = -\omega_c \tau. \quad (12.14)$$

Writing $J_x = -eN_s v_x$ Equations 12.12 and 12.14 can be combined to give

$$R_H \equiv \frac{E_y}{J_x B} = -\frac{1}{N_s e}. \quad (12.15)$$

R_H is of course the *Hall coefficient*. Referring to Figure 11.1, we see that for a two-dimensional system, $J_x = I/w$ and $E_y = V_y/w$, where w is the width of the sample, *i.e.*

$$R_H = \frac{V_y}{IB}. \quad (12.16)$$

As in the case of the three-dimensional systems discussed previously, the low-field Hall coefficient can be used to determine N_s .

In high magnetic fields, $\omega_c \tau \gg 1$. Under these conditions, Equation 12.14 indicates that $|\sigma_{xy}| \gg |\sigma_{xx}|$, so that Equations 12.9 and 12.10 give

$$\rho_{xx} \approx \frac{\sigma_{xx}}{\sigma_{xy}^2} \quad (12.17)$$

and

$$\rho_{xy} \approx \frac{1}{\sigma_{xy}} = R_H B. \quad (12.18)$$

We are therefore left with the rather unexpected result that the resistivity is proportional to the conductivity. This is due to the very important influence of the Hall field, and the way in which resistivity and conductivity have been defined; the resistivity is defined for zero Hall current, and the conductivity for zero Hall field.

At the values of magnetic field corresponding to a completely filled highest Landau level, the “insulating” case, it is the *conductivity* σ_{xx} which goes to zero, and hence the resistivity ρ_{xx} also tends to zero. Physically speaking, what has happened is that, as the probability of scattering goes to zero, an electron (for $E_y = 0$) will go around a cyclotron orbit with a drift in a direction perpendicular to the electric and magnetic fields, and no current will flow along the direction of the applied electric field. For the resistivity (with $J_y = 0$) the Hall field balances out the Lorentz force, and so a current can flow with very small scattering probability, giving a vanishing resistivity. Experimentally, ρ_{xx} can become immeasurably small.

12.3 Quantisation of the Hall resistivity

Some very interesting things happen when we look at the Hall voltage. Experimentally, at the magnetic field values where ρ_{xx} goes to zero, the Hall voltage becomes independent of field, and a *plateau* appears. When ρ_{xx} increases again, then the Hall voltage jumps up to the next plateau. The most interesting feature of this behaviour is the *value* of the Hall resistivity on the plateaux. We may calculate this very easily.

It was mentioned above that ρ_{xx} goes to zero when a whole number of Landau levels are completely filled. At this point the density of electrons is (see Equation 12.4)

$$N_s = j \frac{2eB}{h} \quad (12.19)$$

And therefore the Hall resistivity is given by

$$\rho_{xy} = R_H B = \frac{B}{N_s e} = \frac{1}{j} \frac{h}{2e^2}. \quad (12.20)$$

As we have seen above (see Equation 12.16), for a truly two-dimensional system the width of the sample cancels out (*i.e.* ρ_{xy} is measured in Ω), so that

$$\rho_{xy} \equiv \frac{V_y}{I} = \frac{1}{j} \frac{h}{2e^2} = \frac{12906.5}{j} \Omega. \quad (12.21)$$

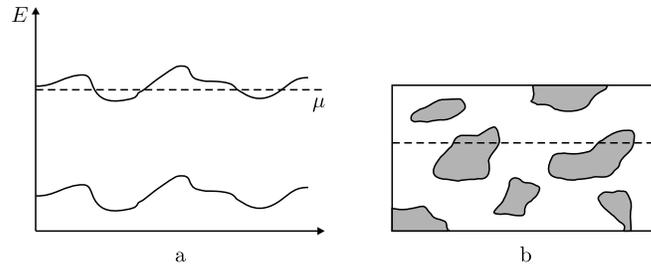


Figure 12.2: (a) Spatial variation of Landau level energies caused by disorder (impurities *etc.*). μ indicates the chemical potential. (b) Plan of the resulting “puddles” of electrons.

The remarkable feature of the experiment is that this value of Hall resistivity, determined only by the fundamental constants h and e , is obtained extremely accurately (see also Section 12.3.2).

However, even though our simple derivation has predicted the value of the quantised resistance, it only gives this value *at one point*, the field at which an integer number of Landau levels are completely filled. In experiments, the plateaux *extend over finite widths of magnetic field*. In order to see why this occurs, we must introduce *disorder*.

12.3.1 Localised and extended states

Figure 12.1 showed broadening of the Landau levels due to scattering of the electrons. One way to look at this is to say that the imperfections in the sample cause a potential energy variation through space, and that the magnitude of this variation is equal to the broadening of the levels. This means that the energy of a Landau level moves up and down as we move through the sample. This is shown in Figure 12.2a.

Let us imagine varying the field. Starting at the field at which one Landau level is filled, the field is lowered. This lowers the Landau level degeneracy so that some electrons must go into the next Landau level. These electrons fill up the lowest states, in the minima of the potential fluctuations. The next step is to draw a two-dimensional map of the occupied states. This is shown in Figure 12.2b. Provided that the field does not change too much, then the occupied states will be in isolated islands, which are not connected to each other. Because they are isolated, these electrons will not play any part in conduction, and the resistivity and Hall voltage from the sample will remain constant, independent of whether these states are occupied or not. Hence we can see the Hall plateaux and zero σ_{xx} (and ρ_{xx}) over a range of field.

We therefore have two classes of state, *extended states* at the centres of Landau levels, which allow the electrons to move through the heterojunction and *localised states*, in the “tails” of the Landau levels, which strand the electrons in isolated puddles. When the μ is in the localised states between Landau level centres, $\sigma_{xx} = \rho_{xx} = 0$ and ρ_{xy} is quantised; when μ is in the extended states close to the Landau level centres, σ_{xx} and hence ρ_{xx} are finite.

12.3.2 A further refinement— spin splitting

When experimental data are examined, the Hall plateaux are actually found at resistances

$$\rho_{xy} \equiv \frac{V_y}{I} = \frac{1}{\nu} \frac{h}{e^2} = \frac{25812.8}{\nu} \Omega, \quad (12.22)$$

(*c.f.* Equation 12.21) where ν is an integer. The problem is that we have ignored the spin-splitting of the Landau levels in the simple treatment above. The Landau level energies are in fact

$$E(l, B) = \frac{\hbar e B}{m^*} \left(l + \frac{1}{2} \right) \pm \frac{1}{2} g^* \mu_B B, \quad (12.23)$$

where μ_B is the Bohr magneton, the \pm takes account of spin-up and spin-down electrons and g^* is the effective g -factor. The resulting Landau-level density of states is shown in Figure 12.3; we now have

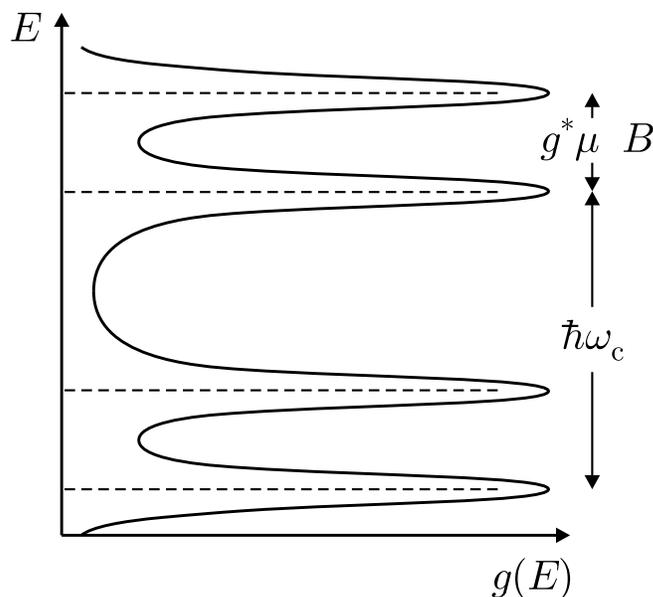


Figure 12.3: Two-dimensional Landau levels in the presence of spin splitting.

twice as many levels as before, each with half the degeneracy, thus explaining the quantised values in Equation 12.22.

The parameter ν is known as the *filling factor*; it is the number of filled (spin-split) half Landau levels, and is given by

$$\nu = \frac{N_s h}{eB}. \quad (12.24)$$

The spin-splitting is much smaller than $\hbar\omega_c$ (e.g. $g^* \sim -0.4$ in GaAs), and so Hall plateaux corresponding to *odd* ν (i.e. those where μ is in a spin gap rather than a cyclotron gap) tend to be resolved only at higher fields.

12.4 Summary

The experimental data in Figure 12.4 summarise the above discussion very nicely. The following points may be noted.

- At integer ν , the chemical potential μ is in the localised states (i.e. away from the centres of the levels). Therefore $\sigma_{xx} = \rho_{xx} = 0$ and $\rho_{xy} = h/\nu e^2$.
- Between integer ν , μ is in the extended states. Therefore σ_{xx} and ρ_{xx} become finite and ρ_{xy} changes.
- As the temperature rises, the plateaux and regions of zero ρ_{xx} become narrower. This is because the Fermi–Dirac distribution function becomes smeared out on either side of μ , allowing some electrons to access extended states even when μ is amongst the localised states.
- Hall plateaux and ρ_{xx} minima with odd values of ν disappear at low fields. This is because odd ν corresponds to μ being in an energy gap $\sim g^* \mu_B B$, much smaller than $\hbar\omega_c$. By contrast, even ν features correspond to μ being in a gap $\sim \hbar\omega_c$.

The Quantum Hall Effect was first discovered in 1980 by von Klitzing *et al.*. Von Klitzing was awarded the 1985 Nobel Physics Prize for its discovery. To date the value of the quantized Hall resistance is known to an accuracy of one part in $\sim 10^8$ and has become the international standard for resistance. It is also the most accurate way to determine the atomic fine-structure constant.¹

¹The atomic fine structure constant is $\alpha = e^2/(4\pi\epsilon_0\hbar c)$; α is linked to h/e^2 only by defined constants.

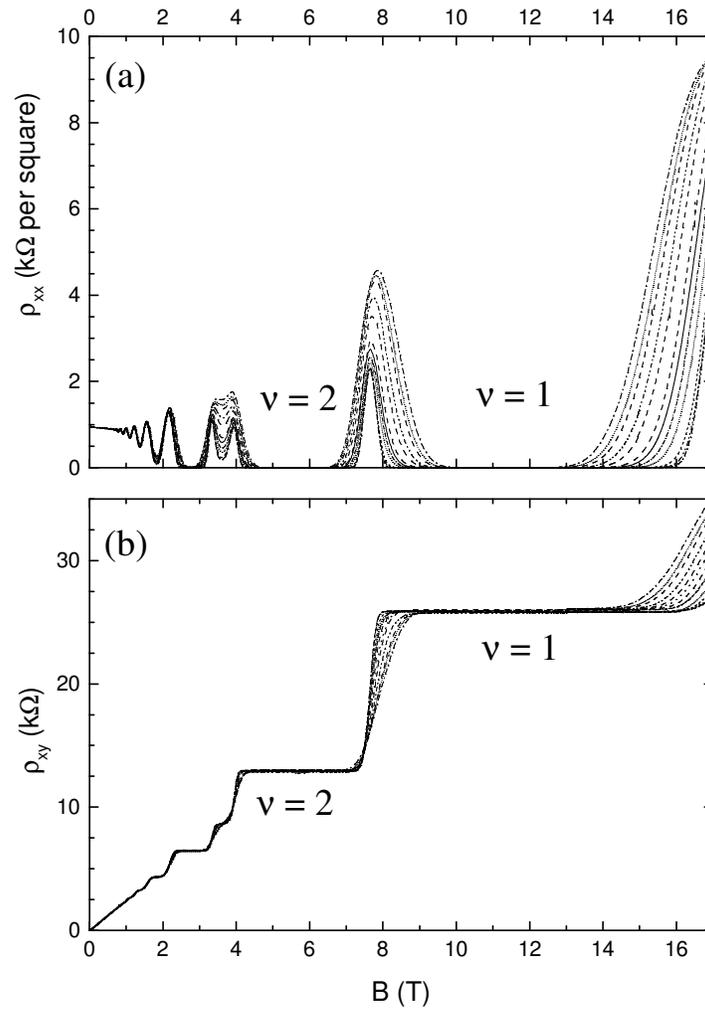


Figure 12.4: Experimental ρ_{xx} and ρ_{xy} data for a GaAs-(Ga,Al)As heterojunction. Data for temperatures between 30 mK and 1.5 K are shown. (Data courtesy of Phil Gee).

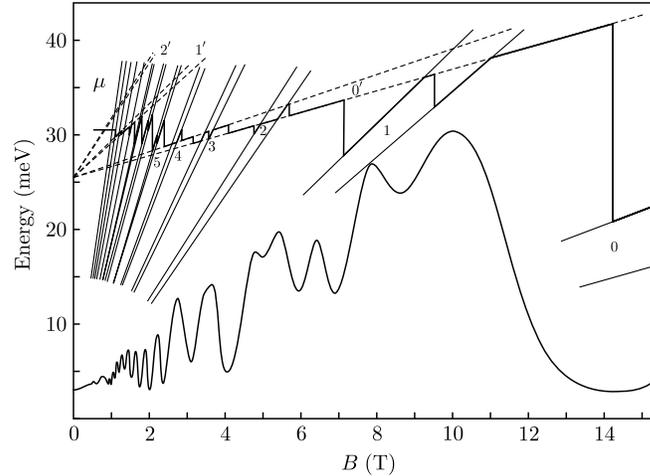


Figure 12.5: ρ_{xx} for a (Ga,In)As-(Al,In)As heterojunction with two populated subbands, plus a schematic representation of the motion of the chemical potential (Fermi energy). (Data from J.C. Portal *et al.*, *Solid State Commun.* **43**, 907 (1982).)

12.5 More than one subband populated

Figure 12.5 shows ρ_{xx} for a (Ga,In)As-(Al,In)As heterojunction with two populated subbands, plus a schematic representation of the motion of the chemical potential (Fermi energy). At low fields, the levels are poorly resolved, and the effects of spin-splitting are not important; two distinct periodicities are observed, given by

$$\Delta(1/B)_1 = \frac{2e}{hN_{s1}}, \quad (12.25)$$

and

$$\Delta(1/B)_2 = \frac{2e}{hN_{s2}}, \quad (12.26)$$

where N_{s1} and N_{s2} are the populations of the two subbands.²

At intermediate fields (~ 10 T), the presence of the second subband complicates the structure of the ρ_{xx} oscillations. However, at high enough fields, the levels will become widely separated in energy and the quantum Hall effect will re-emerge.

12.6 Reading

Simple treatments of the quantum Hall effect are found in *Solid State Physics*, by G. Burns (Academic Press, Boston, 1995) Chapter 18 and *Low-Dimensional Semiconductor Structures*, by M.J. Kelly, (Clarendon Press, Oxford 1995) (spread through the book). This material is a summary of *Band theory and electronic properties of solids*, by John Singleton (Oxford University Press, 2001), Chapter 10, which also describes the *fractional quantum Hall effect*. A good recent review is given in *Fractional quantum Hall effects*, by M. Heiblum and A. Stern, *Physics World*, **13**, no. 3, page 37 (2000)

²This result is a straightforward consequence of the direct proportionality of the cross-sectional area of each subband's two-dimensional Fermi surface to the number of carriers it contains. The discussion of the Shubnikov-de Haas effect in Lecture 9 reminds us that the Fermi-surface area is directly related to the frequency of the oscillations.