Quantum Chromo Dynamics (QCD)

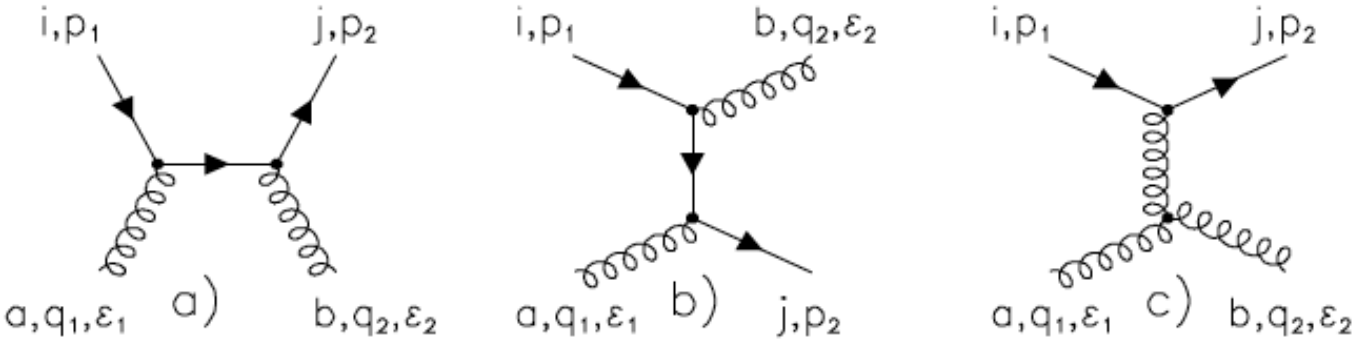
$$\mathcal{L} = i\overline{\Psi}_{iI} \not{D}_{ij} \Psi_{jI} - m_I \overline{\Psi}_I \Psi_I - \frac{1}{2}\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu})$$

Gauge group SU(3)

Gauge bosons Gluons $A_\mu^{a=1..8}$

Fermions Quarks $q_{a=1.,2,3}$ (3 "colours")

Scattering in QCD



$$qg \to qg$$

$$M_s = \frac{-ig^2}{s}\left(T^a_{li}T^b_{jl}\right)\left[\overline{u}^j(p_2)\,\not{\epsilon}_2(\not{p}_1+\not{q}_1)\,\not{\epsilon}_1 u^i(p_1)\right]$$

$$M_u = \frac{-ig^2}{u}\left(T^a_{jk}T^b_{ki}\right)\left[\overline{u}^j(p_2)\,\not{\epsilon}_1(\not{p}_1-\not{q}_2)\,\not{\epsilon}_2 u^i(p_1)\right]$$

$$M_t = \frac{g^2}{t}\left[f_{abc}\left(T^c\right)_{ij}\right] C^{\lambda\mu\nu}(q_1-q_2,-q_1,q_2)\epsilon_{1\mu}\epsilon_{2\nu}\left[\overline{u}^j(p_2)\gamma_\lambda u^i(p_1)\right]$$

Colour factors

$$M_s = \frac{-ig^2}{s}\left(T^a_{li}T^b_{jl}\right)\left[\overline{u}^j(p_2)\,\not{\epsilon}_2(\not{p}_1+\not{q}_1)\,\not{\epsilon}_1 u^i(p_1)\right]$$

$$\bar{u}_j\left(T^bT^a\right)_{ji} Xu_i\left(\bar{u}_j\left(T^bT^a\right)_{ji} Xu_i\right)^\dagger$$
$$\Rightarrow Tr(T^aT^bT^bT^a)$$

$$\left|M_s\right|^2 \rightarrow \quad \frac{1}{3}\frac{1}{8}T^a_{li}T^b_{jl}T^{b*}_{jk}T^{a*}_{ki} = \frac{1}{24}\mathrm{Tr}(T^a\underbrace{T^bT^b}_{(4/3)}T^a) = \frac{2}{9}$$

Q,g colour averages

$C(N)$

$$M_t = \frac{g^2}{t}\left[f_{abc}\left(T_c\right)_{ji}\right]$$
$$\{[\overline{u}(p_2)\,\not{\epsilon}_1 u(p_1)]\,(2q_1-q_2)\epsilon_2 - [\overline{u}(p_2)(\not{q}_1+\not{q}_2)u(p_1)]\,(\epsilon_1\epsilon_2) + [\overline{u}(p_2)\,\not{\epsilon}_2 u(p_1)]\,(2q_2-q_1)\epsilon_1\}$$

$$f_{abc}\bar{u}_jT^cXu_i\left(f_{abd}\bar{u}_jT^dXu_i\right)^\dagger$$
$$\Rightarrow f_{abc}f_{abd}Tr(T^cT^d)$$

$$\left|M_t\right|^2 \rightarrow \quad \frac{1}{3}\frac{1}{8}f_{abc}\left(T^c\right)_{ji}f_{abd}\left(T^d\right)^*_{ji} = \frac{1}{24}\overbrace{f_{abc}f_{abd}}^{3\delta_{cd}}\underbrace{\mathrm{Tr}\left[T^cT^d\right]}_{(1/2)\delta_{cd}} = \frac{1}{2}$$

$C(A)$

$T(N)$

$$M_s = \frac{-ig^2}{s}\left(T^a_{li}T^b_{jl}\right)\left[\overline{u}^j(p_2)\ \not{\epsilon}_2(\not{p}_1+\not{q}_1)\ \not{\epsilon}_1 u^i(p_1)\right]$$

$$M_t = \frac{g^2}{t}\left[f_{abc}\left(T^c\right)_{ij}\right] C^{\lambda\mu\nu}(q_1-q_2,-q_1,q_2)\epsilon_{1\mu}\epsilon_{2\nu}\left[\overline{u}^j(p_2)\gamma_\lambda u^i(p_1)\right]$$

$$f_{abc}\bar{u}_j T^c Xu_i \left(\bar{u}_j\left(T^b T^a\right)_{ji} Xu_i\right)^\dagger$$

$$\Rightarrow f_{abc}Tr(T^cT^aT^b)$$

$$M_t M_s^* \Rightarrow f_{abc}Tr[T^aT^bT^c]$$

$$= \frac{1}{2}f_{abc}Tr[T^a[T^b,T^c]] = \frac{1}{2}f_{abc}f_{bcd}Tr[T^aT^d]$$

$$= \frac{i}{4}f_{abc}f_{abc}$$

$$= C(A)Tr[I]$$

$$= N.N$$

The QCD beta function

$$\begin{aligned}\mathcal{L} = & \tfrac{1}{2} Z_3 A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} + \tfrac{1}{2}\xi^{-1}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu} \\ & - Z_{3g} g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \tfrac{1}{4} Z_{4g} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu \\ & - Z_{2'} \partial^\mu \bar{C}^a \partial_\mu C^a + Z_{1'} g f^{abc} A^c_\mu \partial^\mu \bar{C}^a C^b \\ & + i Z_2 \overline{\Psi}_i \not{\partial} \Psi_i - Z_m m \overline{\Psi}_i \Psi_i + Z_1 g A^a_\mu \overline{\Psi}_i \gamma^\mu T^a_{ij} \Psi_j \ .\end{aligned}$$

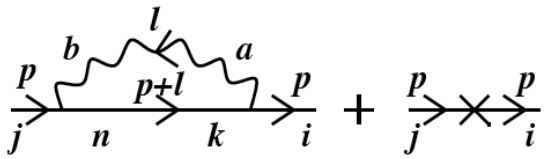
Relation between counter terms:

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_{3g}^2}{Z_3^3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^\varepsilon$$

(Proof: BRST invariance – later)

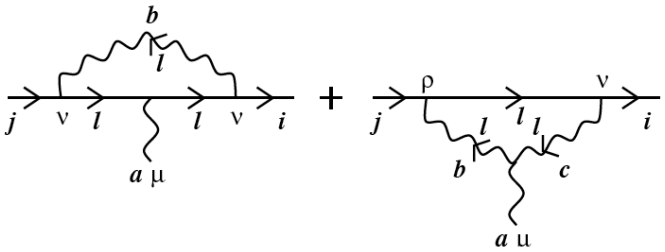
Beta function from quark-quark-gluon coupling

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^{\varepsilon}$$

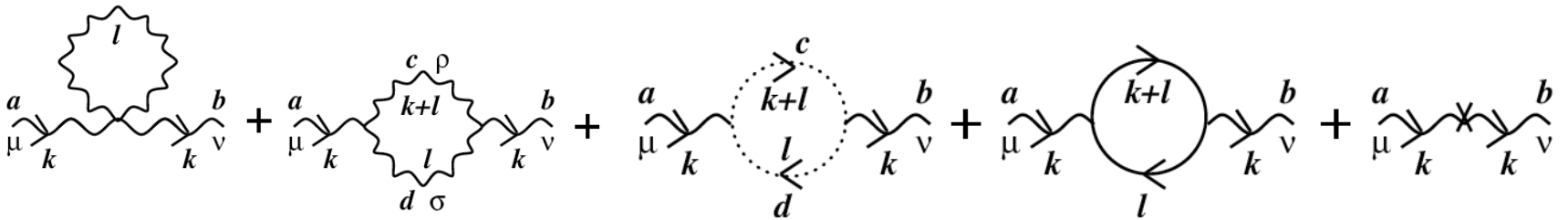


+

Z_2

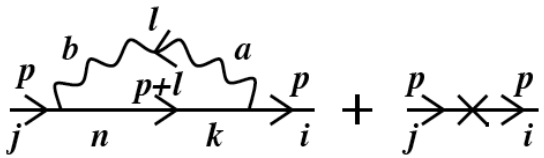


Z_1



Z_3

Z_2



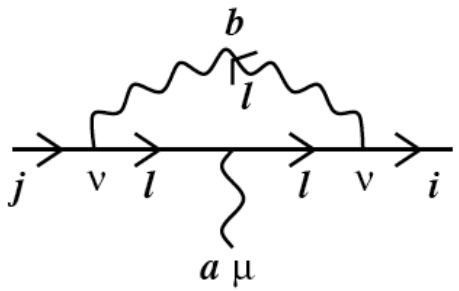
$$(T^a T^a)_{ij} = C(\mathrm{R})\delta_{ij}$$

$$\Sigma_{ij}(\not{p}) = -(g^2/8\pi^2\varepsilon)C(\mathrm{R})\delta_{ij}\not{p} + \text{finite}$$

$$Z_2 = 1 - C(\mathrm{R})\,\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)$$

(Feynman gauge, $\overline{MS}$)

Z_1

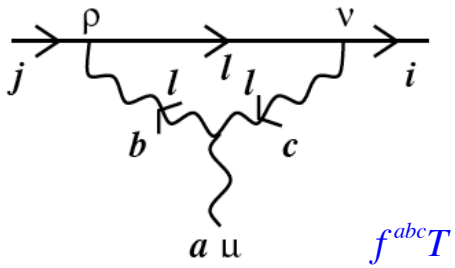


$$
\begin{aligned}
T^b T^a T^b &= T^b\left(T^b T^a + i f^{abc} T^c\right) \\
&= C(\mathrm{R}) T^a + \tfrac{1}{2} i f^{abc} [T^b, T^c] \\
&= C(\mathrm{R}) T^a + \tfrac{1}{2} (i f^{abc})(i f^{bcd}) T^d \\
&= C(\mathrm{R}) T^a - \tfrac{1}{2} (T^a_{\mathrm{A}})^{bc} (T^d_{\mathrm{A}})^{cb} T^d \\
&= \left[C(\mathrm{R}) - \tfrac{1}{2} T(\mathrm{A})\right] T^a \,.
\end{aligned}
$$

from Abelian calc.

$$
i\mathbf{V}^{a\mu}_{ij}(p', p) \;=\; \left[C(\mathrm{R}) - \tfrac{1}{2} T(\mathrm{A})\right] \frac{g^2}{8\pi^2 \varepsilon}\, i g T^a_{ij} \gamma^\mu
$$

Z_1



$f^{abc}T^cT^b = \frac{1}{2}f^{abc}\left[T^c, T^b\right] = \frac{1}{2}if^{abc}f^{cbd}T^d = -\frac{1}{2}iT(A)T^a$

$$(ig)^2 g f^{abc}(T^cT^b)_{ij}\left(\frac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\gamma_\rho(-\not{\ell}+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)}$$
$$\times\,[(\ell-(-\ell))^\mu g^{\nu\rho} + (-\ell-0)^\nu g^{\rho\mu} + (0-\ell)^\rho g^{\mu\nu}]$$

$$N^\mu = \gamma_\rho(-\gamma_\sigma\ell^\sigma + m)\gamma_\nu(2\ell^\mu g^{\nu\rho} - \ell^\nu g^{\rho\mu} - \ell^\rho g^{\mu\nu})$$

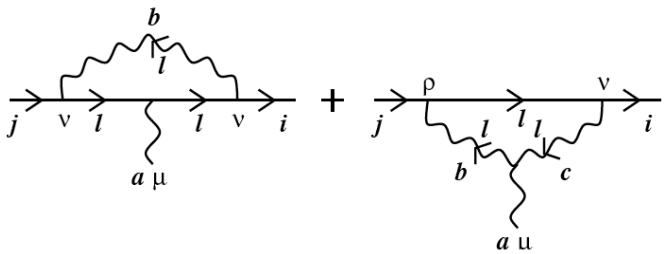
$$\ell^\sigma\ell^\mu \;\longrightarrow\; d^{-1}\ell^2 g^{\sigma\mu}$$

$$N^\mu \to -d^{-1}\ell^2\,(\gamma_\rho\gamma_\sigma\gamma_\nu)(2g^{\sigma\mu}g^{\nu\rho} - g^{\sigma\nu}g^{\rho\mu} - g^{\sigma\rho}g^{\mu\nu})$$
$$\to -d^{-1}\ell^2\,(2\gamma^\nu\gamma^\mu\gamma_\nu - \gamma^\mu\gamma^\nu\gamma_\nu - \gamma_\rho\gamma^\rho\gamma^\mu)$$
$$\to -d^{-1}\ell^2\,(2(d-2) + d + d)\gamma^\mu\ .$$

$i/8\pi^2\varepsilon + O(\varepsilon^0)$

$$\frac{3}{2}T(\mathrm{A})\,g^3\,T^a_{ij}\gamma^\mu \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2(\ell^2+m^2)}$$

Z_1



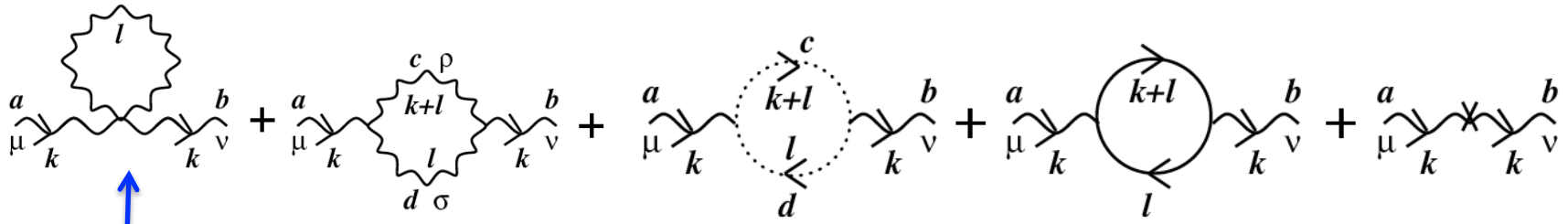
$$\mathbf{V}_{ij}^{a\mu}(0,0)_{\text{div}} = \left(Z_1 + \left[C(\mathrm{R}) - \tfrac{1}{2} T(\mathrm{A}) \right] \frac{g^2}{8\pi^2 \varepsilon} + \tfrac{3}{2} T(\mathrm{A}) \frac{g^2}{8\pi^2 \varepsilon} \right) g T_{ij}^a \gamma^\mu$$

$$Z_1 = 1 - \left[C(\mathrm{R}) + T(\mathrm{A}) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4)$$

$Z_1 \neq Z_2$?

See p442 Srednicki)

Z_3



$$I_1 = \int d^4l \frac{1}{l^2} = 0$$

$$I_2 = \tfrac{1}{2} g^2 f^{acd} f^{bcd} \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{N^{\mu\nu}}{\ell^2 (\ell+k)^2}$$

$$\begin{aligned} N^{\mu\nu} &= [(k+\ell)-(-\ell)]^{\mu} g^{\rho\sigma} + (-\ell-(-k))^{\rho} g^{\sigma\mu} + ((-k)-(k+\ell))^{\sigma} g^{\mu\rho}] \\ &\quad \times [(-k-\ell)-\ell)^{\nu} g_{\rho\sigma} + (\ell-k)_{\rho} \delta_{\sigma}{}^{\nu} + (k-(-k-\ell))_{\sigma} \delta^{\nu}{}_{\rho}] \\ &= -[(2\ell+k)^{\mu} g^{\rho\sigma} - (\ell-k)^{\rho} g^{\sigma\mu} - (\ell+2k)^{\sigma} g^{\mu\rho}] \\ &\quad \times [(2\ell+k)^{\nu} g_{\rho\sigma} - (\ell-k)_{\rho} \delta_{\sigma}{}^{\nu} - (\ell+2k)_{\sigma} \delta^{\nu}{}_{\rho}] \end{aligned} \qquad (73.15)$$

$$f^{acd} f^{bcd} = T(\mathrm{A}) \delta^{ab}$$

$$I_2 = -\tfrac{1}{2} g^2 T(\mathrm{A}) \delta^{ab} \tilde{\mu}^{\varepsilon} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{N^{\mu\nu}}{(q^2 + D)^2}$$

$$D = x(1-x)k^2$$

$$q = \ell + xk$$

$$Z_3 \quad I_2 = -\tfrac{1}{2} g^2 \, T(\mathrm{A}) \delta^{ab} \, \tilde{\mu}^{\varepsilon} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{N^{\mu\nu}}{(q^2 + D)^2}$$

$$D = x(1-x)k^2, \quad q = \ell + xk$$

$$\begin{aligned} N^{\mu\nu} = & -[(2q+(1-2x)k)^{\mu} g^{\rho\sigma} - (q-(1+x)k)^{\rho} g^{\sigma\mu} - (q+(2-x)k)^{\sigma} g^{\mu\rho}] \\ & \times [(2q+(1-2x)k)^{\nu} g_{\rho\sigma} - (q-(1+x)k)_{\rho} \delta_{\sigma}{}^{\nu} - (q+(2-x)k)_{\sigma} \delta^{\nu}{}_{\rho}] \end{aligned}$$

$$\begin{aligned} \rightarrow \; & -2q^2 g^{\mu\nu} - (4d-6) q^{\mu} q^{\nu} \\ & - [(1+x)^2 + (2-x)^2] k^2 g^{\mu\nu} \\ & - [d(1-2x)^2 + 2(1-2x)(1+x) \\ & \quad - 2(2-x)(1+x) - 2(2-x)(1-2x)] k^{\mu} k^{\nu} \end{aligned}$$

(linear terms vanish)

(d=4 for divergent terms)

$$\rightarrow \; -\tfrac{9}{2} q^2 g^{\mu\nu} - (5-2x+2x^2) k^2 g^{\mu\nu} + (2+10x-10x^2) k^{\mu} k^{\nu}$$

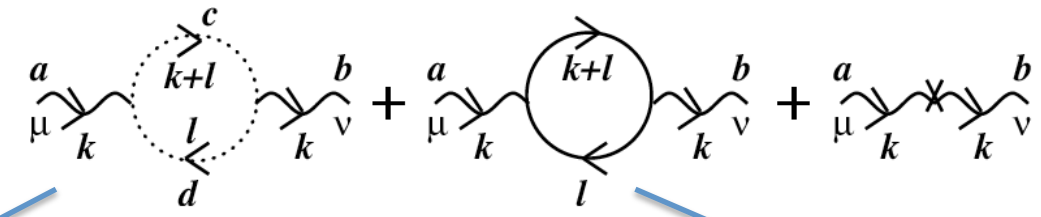
$$\rightarrow \; -(5-11x+11x^2) k^2 g^{\mu\nu} + (2+10x-10x^2) k^{\mu} k^{\nu}$$

$$I_2 = -\frac{i g^2}{16\pi^2} T(\mathrm{A}) \delta^{ab} \frac{1}{\varepsilon} \int_0^1 dx \, N^{\mu\nu} + O(\varepsilon^0)$$

$$= -\frac{i g^2}{16\pi^2} T(\mathrm{A}) \delta^{ab} \frac{1}{\varepsilon} \left(-\tfrac{19}{6} k^2 g^{\mu\nu} + \tfrac{11}{3} k^{\mu} k^{\nu} \right)$$

$$\tilde{\mu}^{\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = \frac{i}{8\pi^2 \varepsilon} + O(\varepsilon^0)$$

Z_3



$$I_3 = (-1)g^2 f^{acd} f^{bdc} \left(\frac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell+k)^\mu \ell^\nu}{\ell^2(\ell+k)^2}$$

$$\mathrm{Tr}(T^a T^b) = T(\mathrm{R})\delta^{ab}$$

$$\begin{aligned}(\ell+k)^\mu \ell^\nu &= (q+(1-x)k)^\mu (q-xk)^\nu \\ &\to \tfrac{1}{4}q^2 g^{\mu\nu} - x(1-x)k^\mu k^\nu \\ &\to -\tfrac{1}{2}x(1-x)k^2 g^{\mu\nu} - x(1-x)k^\mu k^\nu\end{aligned}$$

$$I_3 = -\frac{ig^2}{8\pi^2} T(\mathrm{A})\delta^{ab} \frac{1}{\varepsilon}\left(-\tfrac{1}{12}k^2 g^{\mu\nu} - \tfrac{1}{6}k^\mu k^\nu\right)$$

$$I_4 = -\frac{ig^2}{6\pi^2} n_{\mathrm{F}} T(\mathrm{R})\delta^{ab} \frac{1}{\varepsilon}\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$

$\Rightarrow$

$$\Pi^{\mu\nu ab}(k) = \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu)\delta^{ab}$$

$$\Pi(k^2)_{\mathrm{div}} = -(Z_3-1) + \left[\tfrac{5}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)$$

$$Z_3 = 1 + \left[\tfrac{5}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)$$

$$Z_1 = 1 - \left[C(\mathrm{R}) + T(\mathrm{A})\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \,,$$

$$Z_2 = 1 - C(\mathrm{R}) \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \,,$$

$$Z_3 = 1 + \left[\tfrac{5}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4)$$

(Feynman gauge, $\overline{MS}$)

$$\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3} \alpha \tilde{\mu}^{\varepsilon}$$

$$\alpha \equiv \frac{g^2}{4\pi}$$

$$\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu}$$

$$\ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n}$$

$$G_1(\alpha) = -\left[\tfrac{11}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{\alpha}{2\pi} + O(\alpha^2)$$

$$\frac{d\alpha}{d \ln \mu} = \beta(\alpha) = \alpha^2 G_1(\alpha) = -\left[\frac{11}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\beta(\alpha) = -\left[\tfrac{11}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\dot{\alpha} = \frac{g\dot{g}}{2\pi}$$

$$\frac{d\alpha}{d\ln\mu} = \beta(\alpha) = \alpha^2 G_1(\alpha) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$T(A) = N, T(R) \equiv T(N) = \tfrac{1}{2}$$

$$\beta(\alpha) = -\tfrac{1}{3}(11N - 2n_f)\frac{\alpha^2}{2\pi} \equiv b_0\alpha^2$$

$SU(3)$: For $n_f < 17$ β-function -ve (c.f. QED +ve)

$$\alpha(\mu_1) = \frac{\alpha(\mu_0)}{1 - b_0\alpha(\mu_0)\ln(\frac{\mu_1}{\mu_0})}$$

$\alpha(\mu) \to 0$ as $\mu \to \infty$ Asymptotic Freedom!