The Strong Interaction and LHC phenomenology

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Theoretical Physics Graduate School course

Lecture 5:
The QCD parton model
Deep-Inelastic scattering
Parton Distribution Functions

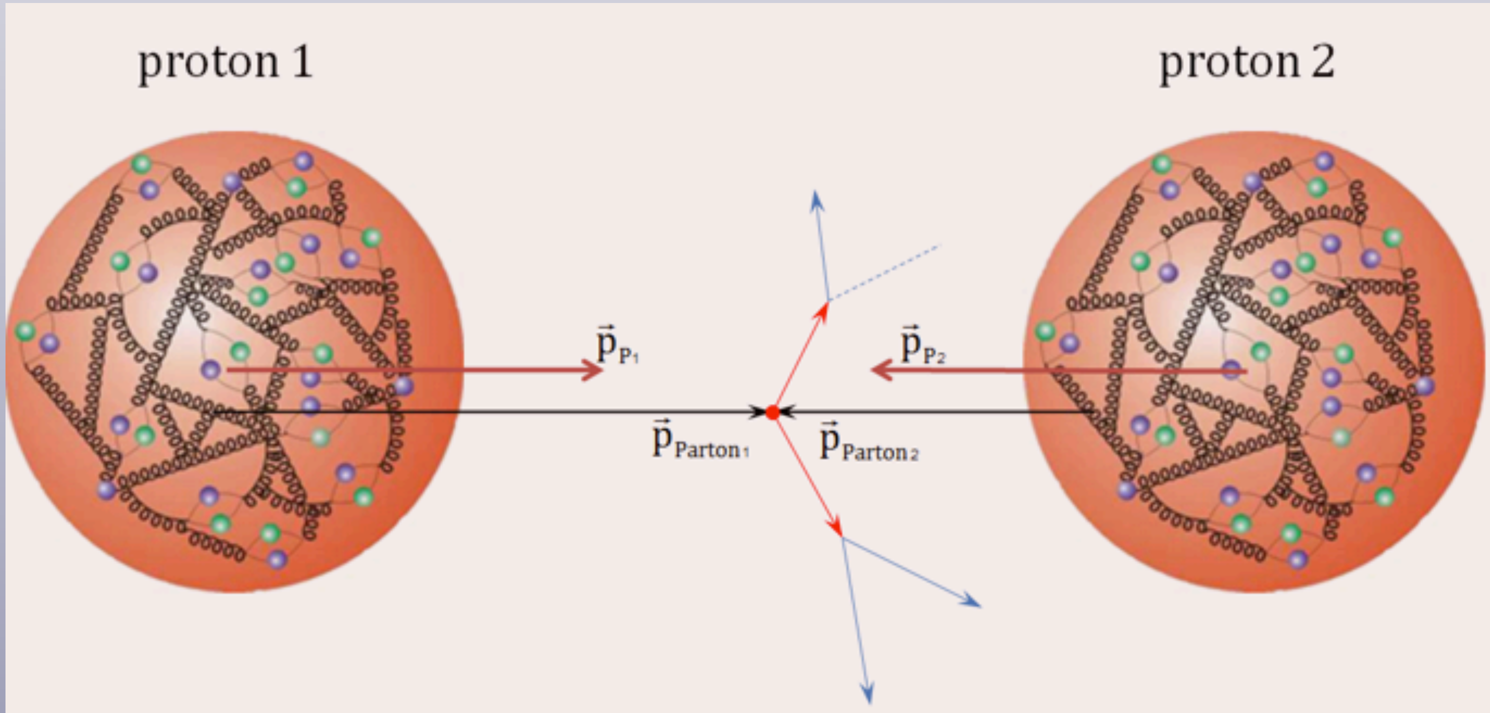
QCD partons in the initial state

In the previous lecture, we have studied **QCD in electron-positron annihilation**, where QCD effects are limited to the **final state**

Now we discuss processes where **one or the two initial state particles are hadrons**

This requires the introduction of important new QCD concepts, related to the fact that **hadrons are not fundamental objects** but composed by QCD partons: quarks and gluons

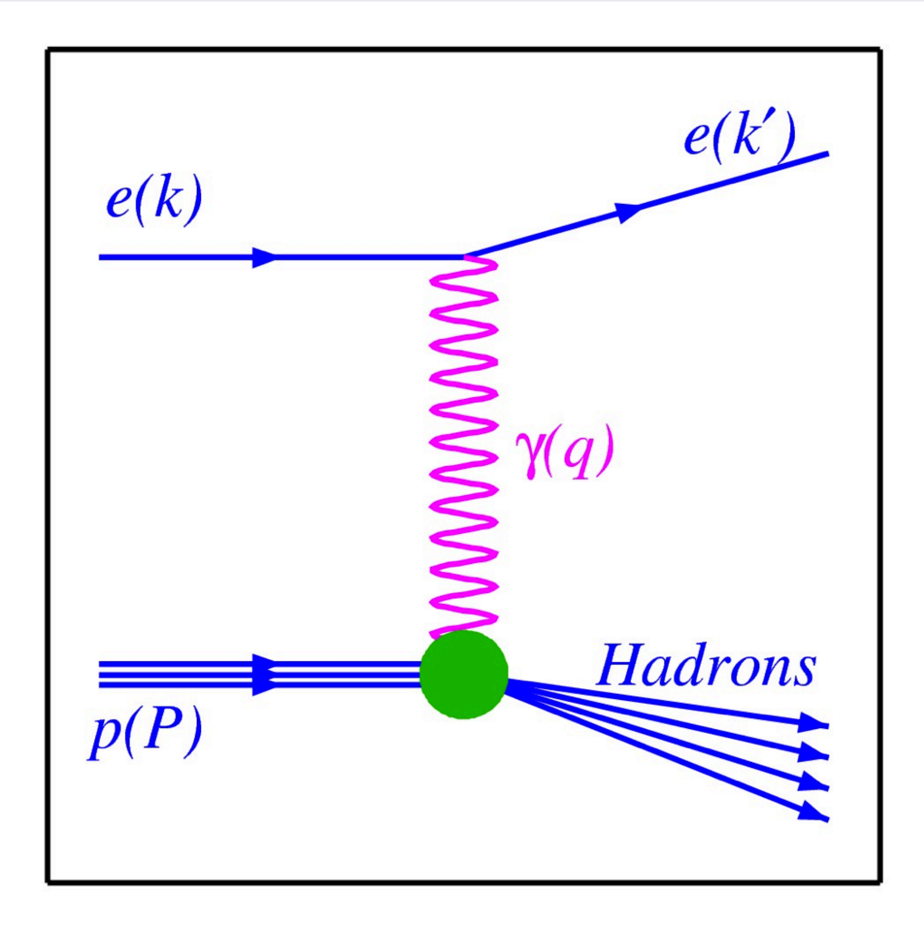
The challenge here is how to derive cross-sections with **quarks and gluons in the initial state** (which we now how to compute using Feynman rules) and cross-sections with initial state hadrons



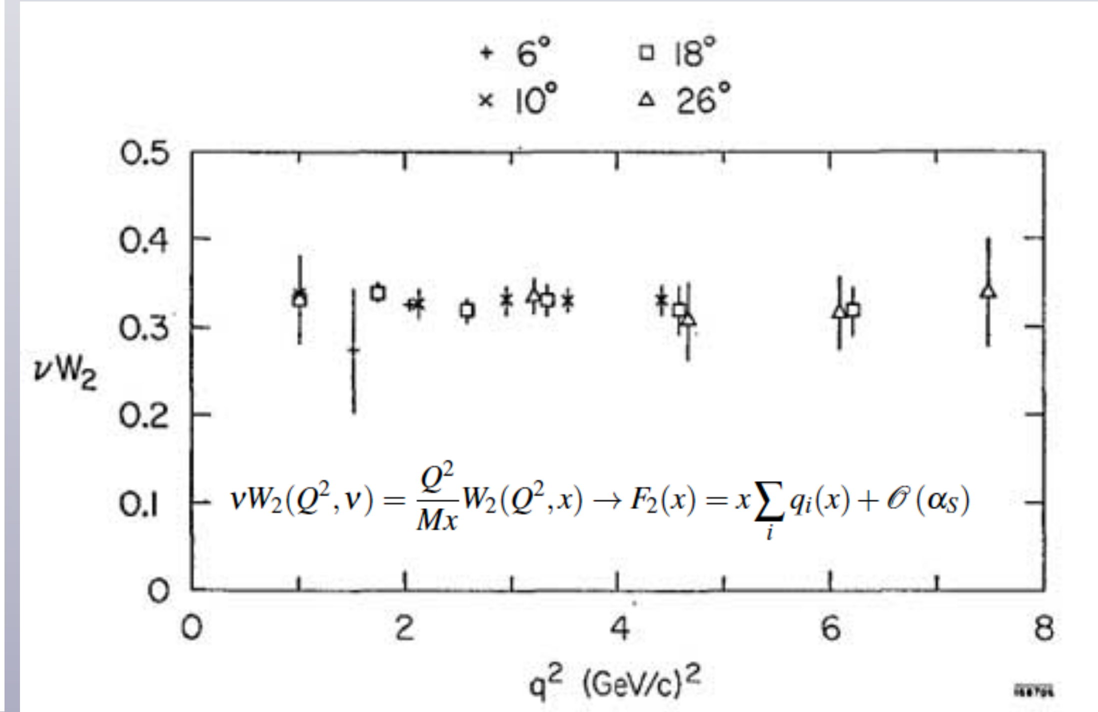
This is a problem of paramount importance to **make sense of LHC collisions**

The QCD parton model

We already discussed the **SLAC deep-inelastic scattering experiment** in the first lecture



$$x_{\rm Bj} = \frac{Q^2}{2\,p\cdot q}, \quad Q^2 = -q^2 \quad y = \frac{q\cdot p}{k\cdot p}$$



For $\mathbf{Q^2} \gg \mathbf{\Lambda}$**,** The DIS structure functions are independent of $\mathbf{Q^2}$

This suggests that in high energy collisions quarks behave as **quasi-free, non-interacting point particles** inside the proton: this is the **QCD parton model**

The QCD parton model

If quarks behave as **quasi-free, non-interacting point particles,** the DIS structure functions should be computed as the **incoherent sum** of all **virtual photon - quark partonic cross-sections**, weighted by the **probability of finding each quark** in the proton with a given **longitudinal momentum fraction**

$$\frac{Q^4 x}{2\pi\alpha^2_{\mathrm{QED}}\,(1+(1-y)^2)}\frac{d^2\sigma^{\mathrm{DIS}}}{dx\,dQ^2} = F_2(x) = \sum_{\mathrm{q},\bar{\mathrm{q}}}\int_x^1 \frac{dz}{z} f_q(z)\,\widehat{\sigma}_{q\gamma^*\to X}\left(\frac{x}{z}\right)$$

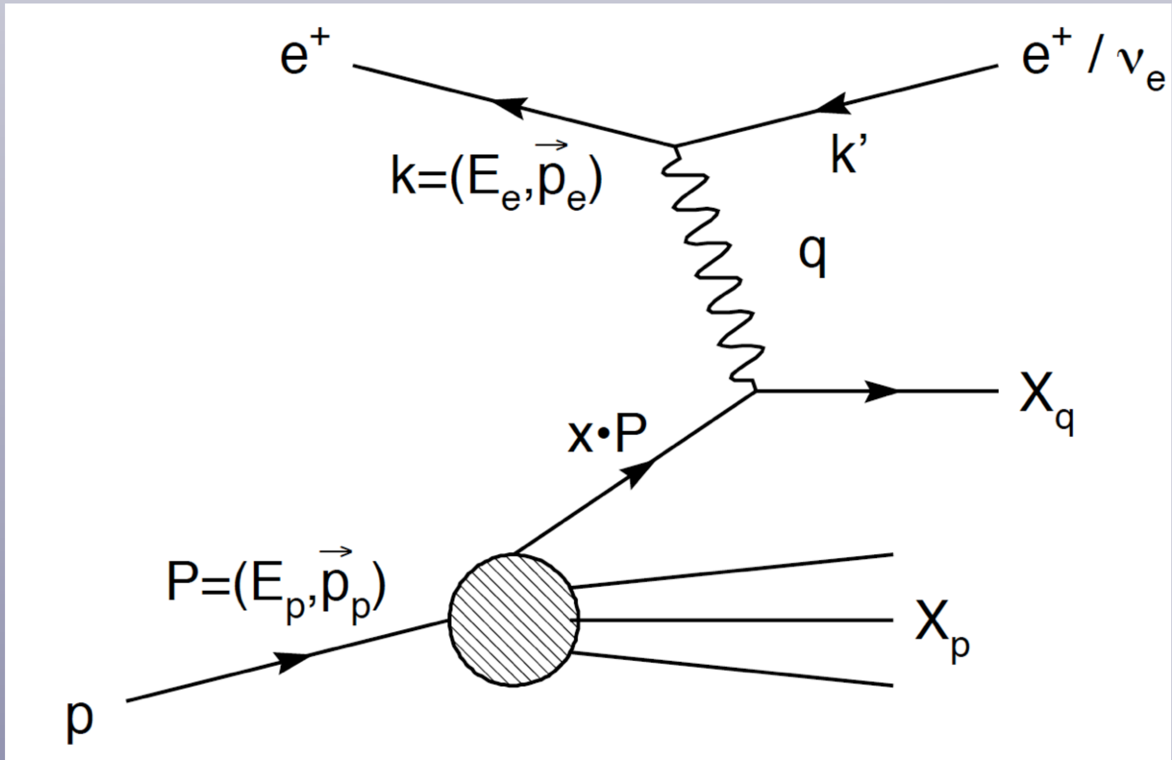
Measured DIS cross-section

"reduced" DIS cross-section = structure function

Sum over all quarks in the proton

Parton Distribution Functions

Perturbative photon-quark cross-section



The **parton distribution functions (PDFs)** are the key ingredient of the QCD parton model

PDFs are non-perturbative, determined by QCD dynamics at the scale $\mathbf{\Lambda_S}$

Therefore, they need to be extracted from experimental hard-scattering data

Parton Distribution Functions

In the parton model, the **parton distribution functions** have probabilistic interpretation: ***$f_q(x)$*** is the probability of finding a quark of **flavor *q*** which carries a **momentum fraction *x*** of the proton longitudinal momentum

From this probabilistic interpretation of series of **PDF sum rules** follows

- The t**otal momentum carried by all the quarks in the proton** must be equal to the proton momentum

$$\sum_{q,\bar{q}} \int_0^1 dx\, x\, f_q(x) = 1$$

Momentum sum rule

- For a given hadron, the **normalization of the PDFs** is such that the valence quantum numbers are reproduced. For a proton, a **uud** bound state, we should have

$$\int_0^1 dx\, (f_u(x) - f_{\bar{u}}) = 2$$
$$\int_0^1 dx\, (f_d(x) - f_{\bar{d}}) = 1$$
$$\int_0^1 dx\, (f_s(x) - f_{\bar{s}}) = 0$$

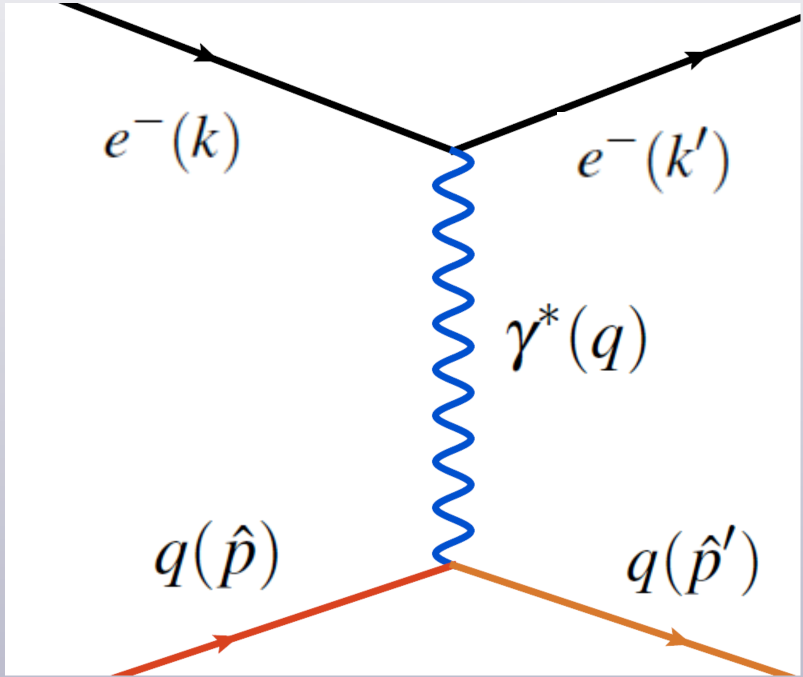
Valence sum rules

As we will see, this **sum rules** are respected also in the presence of QCD radiative corrections

In the following, we derive the explicit expressions of the **DIS structure functions** in the parton model assuming a single quark flavor for simplicity

The QCD parton model

The **partonic cross-section** in **Deep-Inelastic Scattering**, the virtual-photon-quark cross-section can be easily computed using the Feynman rules of QED



$$\mathcal{M} = \frac{iee_q}{q^2}\,\bar{u}(\hat{p}')\gamma^\mu u(\hat{p})\,\bar{u}(k')\gamma_\mu u(k)$$

$$\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}|^2 = \frac{e^4 e_q^2}{4q^2}\,\mathrm{tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right]\,\mathrm{tr}\left[\not{k}'\gamma_\mu\not{k}\gamma_\nu\right]$$

$$\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}|^2 = \frac{8e_q^2e^4}{q^4}\left[(k\cdot p)\,(k'\cdot p') + (k'\cdot p)\,(k\cdot p')\right]$$

Exercise: derive these expressions

It is particularly useful to express the squared matrix element in terms of the **Madelstam variables**

$$\hat{s} \equiv (k+\hat{p})^2 \qquad \hat{t} \equiv (k-k')^2 \qquad \hat{u} \equiv (\hat{p}-k')^2$$

Recall that in the parton model, the incoming quark carries a fraction ξ of the total proton momentum

$$p = \xi\,\hat{p}$$

The QCD parton model

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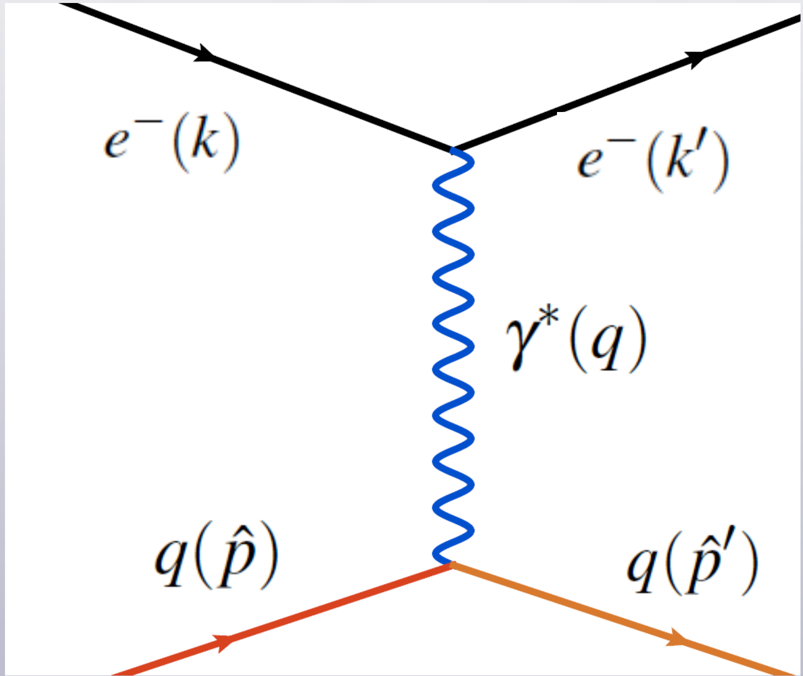


In terms of the Mandelstam variables we find

$$\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}|^2 = 8e_q^2 e^4 \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

Now, using the expression for the 2 -> 2 massless particles scattering cross-section

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2}\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}|^2$$

we end up with the **DIS partonic cross-section**

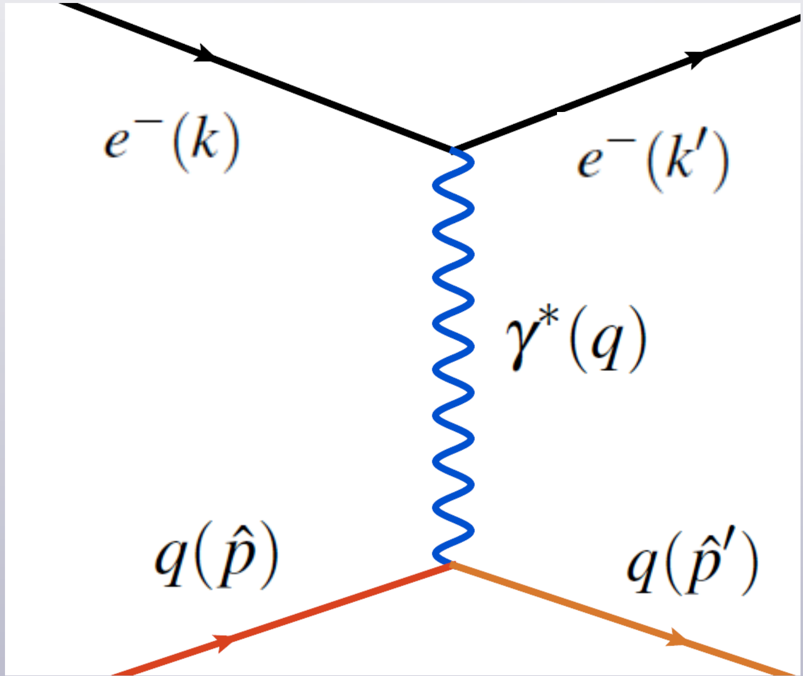
$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{2\pi\alpha^2 e_q^2}{\hat{t}^2}\left(1+\frac{\hat{u}^2}{\hat{s}^2}\right) = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left(1+(1-y)^2\right)$$

where we have used the **basic kinematical variables of deep-inelastic scattering:**

$$Q^2 = -q^2 = -(k-k')^2 \qquad \hat{y} = y = \frac{q\cdot\hat{p}}{k\cdot\hat{p}} = 1-\frac{\hat{u}}{\hat{s}}$$

The QCD parton model

The **partonic cross-section** in **Deep-Inelastic Scattering**, the virtual-photon-quark cross-section can be easily computed using the Feynman rules of QED



$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{2\pi\alpha^2 e_q^2}{\hat{t}^2}\left(1+\frac{\hat{u}^2}{\hat{s}^2}\right) = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left(1+(1-y)^2\right)$$

Now, using the definition of Bjorken-x and the **mass-shell condition for outgoing quark**, which carries a fraction ξ of the total proton momentum:

$$x \equiv \frac{Q^2}{2p\cdot q} \qquad p = \xi\,\hat{p}$$

$$(\hat{p}')^2 = 0 = (\hat{p}+q)^2 \quad \rightarrow \quad Q^2 = 2\hat{p}\cdot q \rightarrow x = \xi$$

Therefore at leading order, the **experimentally accessible Bjorken-x** coincides with the **momentum fraction carried by the struck quark** (but not anymore when radiative corrections are accounted for)

The final expression of the DIS partonic cross-section is

$$\frac{d\hat{\sigma}}{dQ^2dx} = \frac{4\pi\alpha^2}{Q^4}\left(1+(1-y)^2\right)\frac{1}{2}e_q^2\delta(x-\xi)$$

and as required in the parton model, we now add the PDF of the struck quark for the hadronic DIS cross-section

$$\frac{d\sigma}{dQ^2dx} = \frac{4\pi\alpha^2}{Q^4}\left(1+(1-y)^2\right)\frac{1}{2}e_q^2 f_q(x)$$

The QCD parton model

We can now compare with the **experimental definition of the DIS structure functions**, based only on kinematics and Lorentz invariance

From QCD Theory

$$\frac{d\sigma}{dQ^2dx} = \frac{4\pi\alpha^2}{Q^4}\left(1+(1-y)^2\right)\frac{1}{2}e_q^2 f_q(x)$$

From experiment

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1+(1-y^2)\right)F_1(x,Q^2)+\frac{1-y}{x}\left(F_2(x,Q^2)-2xF_1(x,Q^2)\right)\right]$$

and we see that in the parton model, the **DIS structure functions are indeed independent of Q^2:**

Bjorken-scaling, direct evidence of the existence of quarks in the proton

$$\begin{aligned} F_1(x,Q^2) &= \frac{1}{2}e_q^2 f_q(x) \\ F_2(x,Q^2) &= 2xF_1(x,Q^2) \end{aligned}$$

Callan-Gross relation:
direct consequence of spin 1/2 of quarks

When **QCD radiative corrections** are taken into account, Bjorken scaling is violated by **logarithmic corrections**

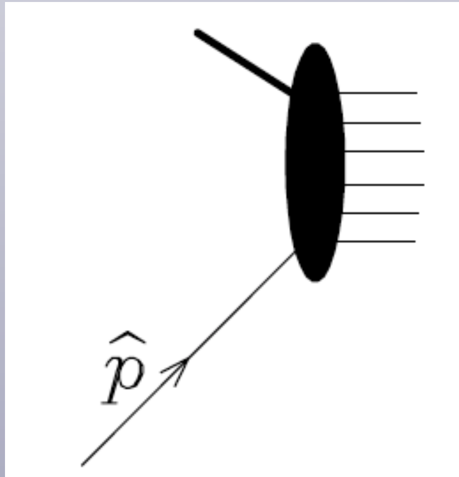
The QCD-improved parton model

Now we want to add **perturbative QCD corrections** to the parton model

In electron-positron annihilation, we saw that adding QCD corrections leads to **infrared divergences,** that can be canceled by suitable definitions of **inclusive final states** (total xsec, jet xsec)

In processes with initial state hadrons, the above also holds, but in addition we have **initial state infrared divergences.**

Let's consider radiative corrections to deep-inelastic scattering



The matrix element will be given by $\mathscr{M}(\hat{p})u(\hat{p})$

where ***M*** represents the rest of the hard scattering

Recall spinor outer product

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p}$$

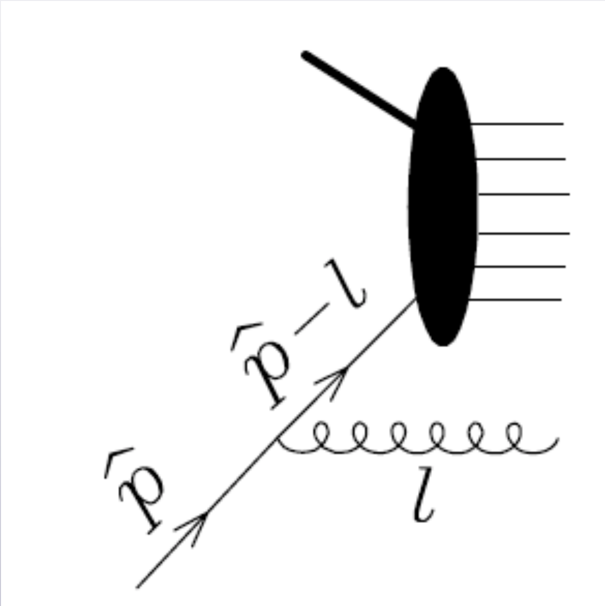
Now the Born cross-section will be

$$\sigma^{\mathrm{born}} = \frac{N}{\hat{p}^0}\frac{1}{2}\sum \mathscr{M}(\hat{p})u(\hat{p})\bar{u}(\hat{p})\mathscr{M}^\dagger(\hat{p}) = \frac{N}{\hat{p}^0}\mathscr{M}(\hat{p})\frac{\hat{\not{p}}}{2}\mathscr{M}^\dagger(\hat{p})$$

Let's study next the impact of radiative corrections

We will consider **gluon radiation** off the initial-state incoming quark

The QCD-improved parton model



Now the matrix element reads

$$g_s \mathscr{M}(\hat{p}-l)\,\frac{\hat{\not p}-\not l}{(\hat{p}-l)^2}\,t^a \varepsilon_a^\mu(l)\gamma_\mu u(\hat{p})$$

We know from **e+e-** that when the gluon is **collinear** to the quark there will be a singularity. Then it is useful to parametrize ***l*** as

$$l=(1-z)\hat{p}+l_\perp+\xi\eta$$

where η is an arbitrary massless vector perpendicular to $l_\perp$

Now in this basis the gluon phase-space factor is

$$\frac{d^3 l}{2l^0(2\pi)^3}=\frac{d^2 l_\perp}{2\,(2\pi)^3}\,\frac{dz}{1-z}$$

With these ingredients we can determine the most singular part of the total cross-section, the collinear limit. After some algebra we find

$$|\mathscr{M}_1|^2=g_S^2\frac{2}{|l_\perp^2|}(1+z^2)\mathscr{M}(\hat{p}-l)\frac{\not p}{2}\mathscr{M}^\dagger(\hat{p}-l)$$

The QCD-improved parton model

Adding the flux factor and integrating over all the phase space for gluon emission we end up with the **real emission cross-section**

$$\sigma^{(1)} = \frac{\alpha_S C_F}{2\pi} \int \sigma^{(0)}(zp) \frac{1+z^2}{1-z} \frac{dl_\perp^2}{l_\perp^2} dz$$

Exercise: derive this cross-section. Recall that we are working in the collinear limit, where $l \simeq (1-z)\,\hat{p}$

As in the case of e+e-, **real emission corrections factorize from the Born term** in the suitable **infrared limit**

Including also virtual corrections, the full NLO cross-section in **Deep-Inelastic Scattering** reads

$$\sigma^{(1)} = \frac{\alpha_S C_F}{2\pi} \int \left[\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p})\right] \frac{1+z^2}{1-z} \frac{dl_\perp^2}{l_\perp^2} dz$$

From this result, there are two important consequences

- When the gluon is very soft, **z -> 0,** the real emission cross-section has a **soft singularity,** which is **cancelled out** by virtual corrections
- The real emission cross-section also has a **collinear singularity, $l_\perp$ -> 0,** which does **not** cancel out

Therefore, infrared divergences do not cancel for **initial state QCD radiation :** failure of parton model

The QCD-improved parton model

We know show how to improve the naive parton model and **avoid collinear divergences**. From the NLO cross-section, it is reasonable to define a **splitting function** so that we can write

$$\sigma^{(1)} = \frac{\alpha_S C_F}{2\pi} \int \left[\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p})\right] \frac{1+z^2}{1-z} \frac{dl_\perp^2}{l_\perp^2} dz = \frac{\alpha_S C_F}{2\pi} \int P_{qq}(z) \sigma^{(0)}(z\hat{p}) \frac{dl_\perp^2}{l_\perp^2} dz$$

and despite the name, **splitting functions** are actually **distributions** defined by the **plus prescription**

$$P_{qq}(z) \equiv \left(\frac{1+z^2}{1-z}\right)_+ \qquad \int_0^1 \left(\frac{1+z^2}{1-z}\right)_+ f(z) \equiv \int_0^1 \left(\frac{1+z^2}{1-z}\right) (f(z) - f(1))$$

Splitting functions are ubiquitous in perturbative QCD: their **universal character** tells us that they will appear in any process which **involves collinear splittings**, both initial state and final state

To continue our calculation, it is useful to introduce the following notation for **convolutions**

$$(f_1(x) \otimes f_2(x) \otimes \ldots \otimes f_n(x))\, \sigma(p) \equiv \int \Pi_{i=1}^n \left(dx_i f(x_i)\right) \sigma\left(x_1 \ldots x_n p\right)$$

It is easy to the that **convolutions** are commutative and that the **identity operator is δ(1- x)**

The QCD-improved parton model

We can now take the NLO cross-section, perform the integral in $\mathbf{l}_\perp$ with some **cut-off λ** and add the Born

$$\sigma^{(1)} = \frac{\alpha_S C_F}{2\pi} \int \left[\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p})\right] \frac{1+z^2}{1-z} \frac{dl_\perp^2}{l_\perp^2} dz = \frac{\alpha_S C_F}{2\pi} \int P_{qq}(z) \sigma^{(0)}(z\hat{p}) \frac{dl_\perp^2}{l_\perp^2} dz$$

and we get that the total cross-section can be written in the **convolution notation** as

$$\sigma^{\mathrm{NLO}} = \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{Q^2}{\lambda^2} P_{qq}\right) \sigma^{(0)}(\hat{p})$$

The cutoff λ is an infrared scale (quark mass, confinement effect) that **physically regulates the collinear divergence**

where $\mathbf{Q^2}$ is the maximum kinematically allowed value of $\mathbf{l}_\perp$

The arbitrary scale μ is known as the **factorization scale**, not to mix with the renormalization scale

Now we can write the NLO cross-section, throwing away higher order terms, as

$$\sigma^{\mathrm{NLO}} = \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{\mu^2}{\lambda^2} P_{qq}\right) \otimes \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{Q^2}{\mu^2} P_{qq}\right) \sigma^{(0)}(\hat{p})$$

Now we can go back to the parton model formulae, and remember that there **partonic hard-scattering cross-sections** appear always convoluted with **non-perturbative parton distributions**

$$\frac{Q^4 x}{2\pi\alpha_{\mathrm{QED}}^2 (1+(1-y)^2)} \frac{d^2\sigma^{\mathrm{DIS}}}{dx dQ^2} = F_2(x) = \sum_{q,\bar{q}} \int_x^1 \frac{dz}{z} f_q(z) \widehat{\sigma}_{q\gamma^* \to X}\left(\frac{x}{z}\right)$$

The QCD-improved parton model

In the convolution notation, and with for an individual quark, the **DIS cross-section reads**

$$\sigma^{\rm DIS}(p) = f_q\, \sigma^{NLO}(p) = \int dx\, f_q(x)\, \sigma^{NLO}(\hat{p} = xp)$$

Hadron-level cross-section

proton momentum

quark PDF

Parton level cross-section
Quark carries fraction x of proton's momentum

The NLO parton level cross-section has a **collinear divergence**, which can be regulated by **low-energy, non-perturbative physics** such as quark masses, confinement etc

The parton distributions themselves are also determined by **low-scale, non-perturbative dynamics**

The NLO hadron level cross-section can be rendered finite if the **PDFs are redefined** to absorb the **collinear singularities**, which makes physical sense since both are **infrared phenomena**

$$\sigma^{\rm DIS}(p) = f_q\, \sigma^{NLO}(\hat{p}) = \tilde{f}_q(\mu)\, \hat{\sigma}(p,\mu)$$

$$\tilde{f}(\mu) \equiv f \otimes \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{\mu^2}{\lambda^2} P_{qq} \right)$$

$$\hat{\sigma}(p,\mu) \equiv \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{Q^2}{\mu^2} P_{qq} \right) \sigma^{(0)}(\hat{p})$$

QCD Factorization

$$\sigma^{\rm DIS}(p) = f_q \, \sigma^{NLO}(\hat{p}) = \tilde{f}_q(\mu)\hat{\sigma}(p,\mu)$$

$$\tilde{f}(\mu) \equiv f \otimes \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{\mu^2}{\lambda^2} P_{qq} \right) \qquad \hat{\sigma}(p,\mu) \equiv \left(\mathbb{1} + \frac{\alpha_s C_F}{2\pi} \ln \frac{Q^2}{\mu^2} P_{qq} \right) \sigma^{(0)}(\hat{p})$$

Now the NLO hadron-level DIS cross-section is finite, and

- ☑ The non-perturbative PDFs now depend on an arbitrary scale, called the **factorization scale**
- ☑ The **QCD-improved PDFs** are still non-perturbative, but their dependence with the **factorization scale** is determined by **perturbative QCD**
- ☑ The parton-level cross-section also depend now on the **factorization scale**
- ☑ It is important not to mix the **renormalization scale** (which arises from the subtraction of UV divergences in the renormalization procedure) with the **factorization scale** (from the subtraction of infrared collinear singularities in processes with **initial state hadrons**)

This redefinition of the PDFs is **universal** and **process-independent**, and holds to all orders in perturbation theory, as proved by the **QCD Factorization Theorems**

PDF evolution equations

With the information we have obtained, we can determine the **evolution equations for the PDFs** that indicate their dependence on the **factorization scale**

If we differentiate wrt the factorization scale we get

$$\sigma^{\mathrm{DIS}}(p) = f_q \sigma^{NLO}(\hat{p}) = \tilde{f}_q(\mu)\hat{\sigma}(p,\mu)$$

$$\mu^2 \frac{\partial}{\partial \mu^2}\sigma^{\mathrm{DIS}}(p) = 0 = \left[\mu^2 \frac{\partial}{\partial \mu^2}\tilde{f}_q(\mu)\right]\hat{\sigma}(p,\mu) + \tilde{f}_q(\mu)\left[\mu^2 \frac{\partial}{\partial \mu^2}\hat{\sigma}(p,\mu)\right]$$

Keeping only the leading order and using the fact that the initial process is arbitrary, we get (**exercise**)

$$\mu^2 \frac{\partial}{\partial \mu^2}\tilde{f}_q(\mu) = \frac{\alpha_s C_F}{2\pi}\tilde{f}_q(\mu) \otimes P_{qq}$$

Non-singlet evolution equation (only quarks)
Including gluons leads to the **singlet** evolution equations

which is the PDF evolution equation for processes which do not involve gluons in the initial state

Once we obtain a **determination of the PDFs** at some given scale, their **evolution to other scales** is determined by perturbative QCD using these evolution equations

For instance, **measure PDFs in DIS at Q = 10 GeV**, and use the **evolution equations** to predict **LHC cross-sections** at Q = 1 TeV

PDF evolution equations

Taking into account also gluons, we end up with the full set of PDF evolution equations, the so called **DGLAP** (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) evolution equations

These equations are a set of **N_{flav} +1 coupled** integro-differential equations

$$\mu^2 \frac{\partial}{\partial \mu^2} f_i(\mu) = \sum_j P_{ij} \otimes f_j(\mu) \, ,$$

$$P_{ij}(y) = \frac{\alpha_S(\mu)}{2\pi} P_{ij}^{(0)}(y) + \left(\frac{\alpha_S(\mu)}{2\pi} \right)^2 P_{ij}^{(1)}(y) + \ldots$$

$$P_{qq}^{(0)}(x) = P_{\bar{q}\bar{q}}^{(0)}(x) = C_{\mathrm{F}} \left(\frac{1+x^2}{1-x} \right)_+ \, ,$$

$$P_{qg}^{(0)}(x) = P_{\bar{q}g}^{(0)}(x) = T_{\mathrm{f}} \left(x^2 + (1-x)^2 \right) \, ,$$

$$P_{gq}^{(0)}(x) = P_{g\bar{q}}^{(0)}(x) = C_{\mathrm{F}} \frac{1+(1-x)^2}{x} \, ,$$

$$P_{gg}^{(0)}(x) = 2C_{\mathrm{A}} \left[z \left(\frac{1}{1-z} \right)_+ + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{n_{\mathrm{f}}}{6C_{\mathrm{A}}} \right) \delta(1-x) \right]$$

A variety of numerical and semi-analytical methods exist for **efficient solutions of the DGLAP equations**

QCD splitting functions

The **QCD splitting functions** have been computed up to **NNLO in the strong coupling**

Their **complexity** increases dramatically each perturbative order

Non-singlet splitting functions at NLO

$$\gamma_{\rm ns}^{(1)+}(N) = 4C_A C_F\Big(2\mathbf{N}_+ S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3}S_1 + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{151}{18}S_1 + 2S_{1,-2} - \frac{11}{6}S_2\Big]\Big)$$
$$+4C_F n_f\Big(\frac{1}{12} + \frac{4}{3}S_1 - (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{11}{9}S_1 - \frac{1}{3}S_2\Big]\Big) + 4C_F{}^2\Big(4S_{-3} + 2S_1 + 2S_2 - \frac{3}{8}$$
$$+\mathbf{N}_-\Big[S_2 + 2S_3\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1 + 4S_{1,-2} + 2S_{1,2} + 2S_{2,1} + S_3\Big]\Big)\,, \qquad (3.5)$$

$$\gamma_{\rm ns}^{(1)-}(N) = \gamma_{\rm ns}^{(1)+}(N) + 16C_F\Big(C_F - \frac{C_A}{2}\Big)\Big((\mathbf{N}_- - \mathbf{N}_+)\Big[S_2 - S_3\Big] - 2(\mathbf{N}_- + \mathbf{N}_+ - 2)S_1\Big)\,. \qquad (3.6)$$

QCD splitting functions

The **QCD splitting functions** have been computed up to **NNLO in the strong coupling**

Their **complexity** increases dramatically each perturbative order

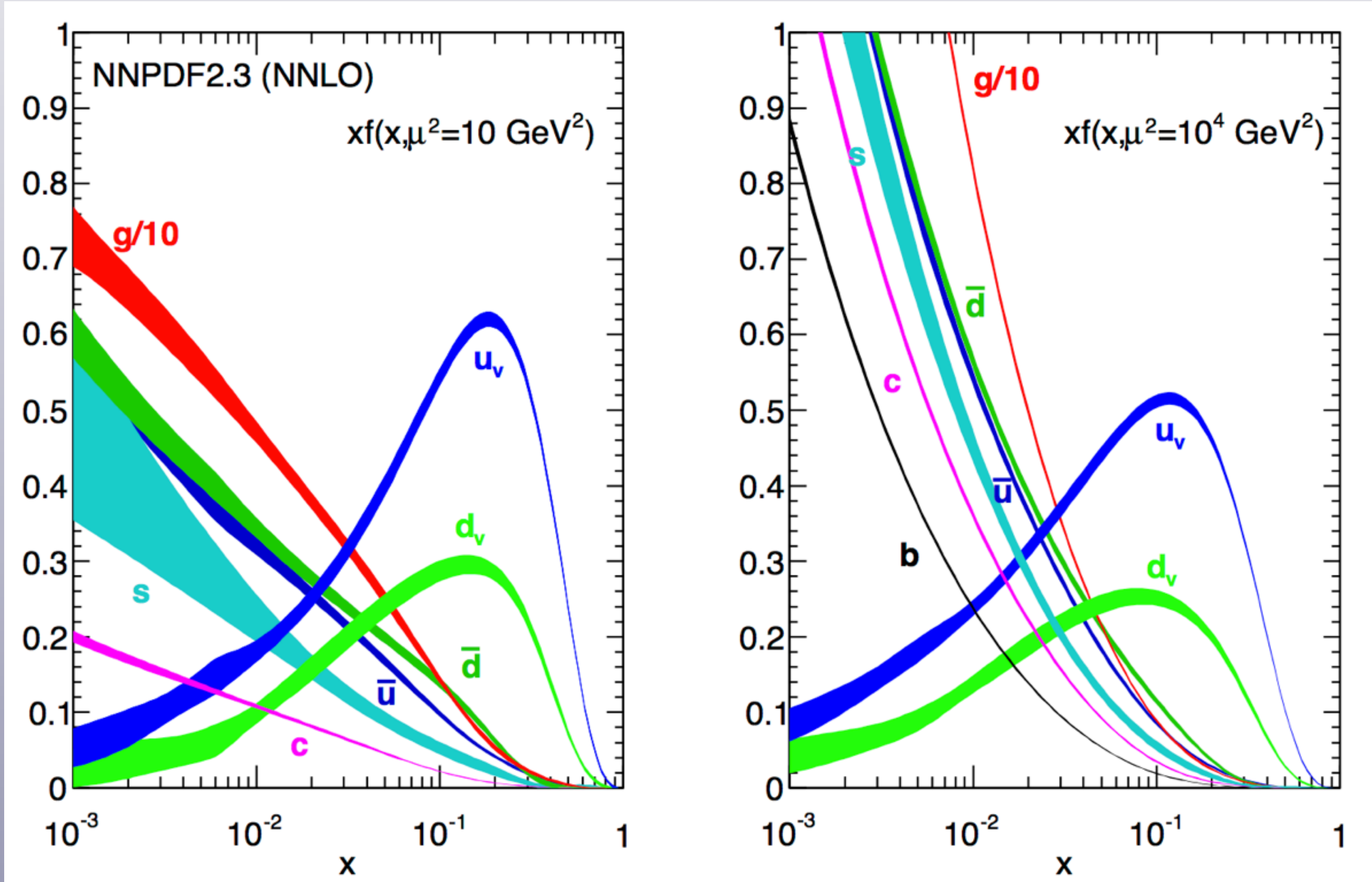
Non-singlet splitting functions at **NNLO** (continue for two more pages ...)

$$
\begin{aligned}
\gamma_{\rm ns}^{(2)+}(N) \; = \; & 16 C_A C_F n_f \Big(\frac{3}{2}\zeta_3 - \frac{5}{4} + \frac{10}{9}S_{-3} - \frac{10}{9}S_3 + \frac{4}{3}S_{1,-2} - \frac{2}{3}S_{-4} + 2S_{1,1} - \frac{25}{9}S_2 \\
& + \frac{257}{27}S_1 - \frac{2}{3}S_{-3,1} - \mathbf{N}_+ \Big[S_{2,1} - \frac{2}{3}S_{3,1} - \frac{2}{3}S_4 \Big] - (\mathbf{N}_+ - 1)\Big[\frac{23}{18}S_3 - S_2 \Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_{1,1} \\
& + \frac{1237}{216}S_1 + \frac{11}{18}S_3 - \frac{317}{108}S_2 + \frac{16}{9}S_{1,-2} - \frac{2}{3}S_{1,-2,1} - \frac{1}{3}S_{1,-3} - \frac{1}{2}S_{1,3} - \frac{1}{2}S_{2,1} - \frac{1}{3}S_{2,-2} + S_1\zeta_3 \\
& + \frac{1}{2}S_{3,1} \Big] \Big) + 16 C_F {C_A}^2 \Big(\frac{1657}{576} - \frac{15}{4}\zeta_3 + 2S_{-5} + \frac{31}{6}S_{-4} - 4S_{-4,1} - \frac{67}{9}S_{-3} + 2S_{-3,-2} \\
& + \frac{11}{3}S_{-3,1} + \frac{3}{2}S_{-2} - 6S_{-2}\zeta_3 - 2S_{-2,-3} + 3S_{-2,-2} - 4S_{-2,-2,1} + 8S_{-2,1,-2} - \frac{1883}{54}S_1 \\
& - 10S_{1,-3} - \frac{16}{3}S_{1,-2} + 12S_{1,-2,1} + 4S_{1,3} - 4S_{2,-2} - \frac{5}{2}S_4 + \frac{1}{2}S_5 + \frac{176}{9}S_2 + \frac{13}{3}S_3 \\
& + (\mathbf{N}_- + \mathbf{N}_+ - 2)\Big[3S_1\zeta_3 + 11S_{1,1} - 4S_{1,1,-2} \Big] + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{9737}{432}S_1 - 3S_{1,-4} + \frac{19}{6}S_{1,-3} \\
& + 8S_{1,-3,1} + \frac{91}{9}S_{1,-2} - 6S_{1,-2,-2} - \frac{29}{3}S_{1,-2,1} + 8S_{1,1,-3} - 16S_{1,1,-2,1} - 4S_{1,1,3} - \frac{19}{4}S_{1,3} \\
& + 4S_{1,3,1} + 3S_{1,4} + 8S_{2,-2,1} + 2S_{2,3} - S_{3,-2} + \frac{11}{12}S_{3,1} - S_{4,1} - 4S_{2,-3} + \frac{1}{6}S_{2,-2} - \frac{1967}{216}S_2
\end{aligned}
$$

PDF evolution

The **DGLAP evolution of PDFs** is a key ingredient for LHC phenomenology: it allows us to measure PDFs in some process at some scale (say DIS) and the evolve upwards to make **LHC predictions**

Measure PDFs at 10 GeV2 **Evolve in Q2 and make LHC predictions**



Different PDF combinations evolve in a different way, for instance, gluon and sea quarks much steeper at small-x than valence quarks

Particle Data Group Review, 2013 update

PDF Sum Rules in pQCD

We saw that in the **naive parton model,** where PDFs have a **probabilistic interpretation**, a series of sum rules but be obeyed by the PDFs: **momentum and valence sum rules**

With perturbative QCD effects, the naive picture is not valid anymore, and PDFs lose their probabilistic interpretation. However, since sum rules arise from **basic conservation principles** (energy conservation and quark flavor conservation), they should also be valid in this case. And indeed they are

For instance, for the valence sum rule, starting from the **non-singlet** DGLAP equation, we have

$$\mu^2 \frac{\partial f_q(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z) f_q\left(\frac{x}{z},\mu^2\right)$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dx \int_x^1 \frac{dz}{z} P_{qq}(z) \left[f_u\left(\frac{x}{z},\mu^2\right) - f_{\bar{u}}\left(\frac{x}{z},\mu^2\right)\right]$$

$$= \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dz \int_0^1 dy P_{qq}(z) \left[f_u\left(y,\mu^2\right) - f_{\bar{u}}\left(y,\mu^2\right)\right] = 0$$

where we have used

$$\int_0^1 dx \int_x^1 dz = \int_0^1 dz \int_0^z dx \qquad y \equiv \frac{x}{z} \qquad \int_0^1 dz P_{qq}(z) = 0$$

Therefore, the **PDF sum rules** in the parton model are also respected in **perturbative QCD**

Exercise: check that the **momentum sum rule** is also respected by the DGLAP evolution equations

Summary

In this lecture we have learnt that:

☑ The **QCD parton model**, where hadrons are treated as **bound states of semi-free quarks and gluons**, is extremely successful to explain the deep-inelastic scattering measurements

☑ In this model, the probability to find **quarks and gluons with a given fraction of the total hadron momentum** is determined by the non-perturbative **Parton Distribution Functions**

☑ Improving the parton model with **perturbative QCD** corrections, we found uncancelled **collinear divergences** that can be absorbed into a redefinition of the PDFs

☑ This leads to a system of coupled evolution equations for the **scale dependence of the PDFs**, the DGLAP evolution equations

☑ The QCD factorization theorem states that **PDFs are universal, process-independent quantities**: they can be measured in some process (say deep-inelastic scattering) and then used to derive predictions for some other processes (like hadron collisions at the LHC)

☑ We will discuss later in these lectures the strategy that needs to be used to extract PDFs from experimental data: the **global QCD analysis**