QFT 2: Problem Set 1

Due date: Friday of 5th week (15th of February)

Discussion: Fisher room, 4 pm,
Monday of 6th week (18th of February)

Problem 1 *Grassmann numbers*

Evaluate the following Grassmann integrals

$$\int d^N\theta^* \, d^N\theta \, \times e^{-\vec{\theta}^\dagger \vec{\theta}} = 1, \tag{1}$$

$$\int d^N\theta^* \, d^N\theta \, \times e^{-\vec{\theta}^\dagger A \vec{\theta}} = \det(A), \tag{2}$$

$$\int d^N\theta^* \, d^N\theta \, \times \theta_k \, \theta_l^* \, e^{-\vec{\theta}^\dagger A \vec{\theta}} = \det(A) \left(A^{-1}\right)_{kl}, \tag{3}$$

$$\int d^N\theta^* \, d^N\theta \, \times e^{-\vec{\theta}^\dagger A \vec{\theta} + \vec{K}^\dagger \vec{\theta} + \vec{\theta}^\dagger \vec{K}} = \det(A) \, e^{\vec{K}^\dagger A^{-1} \vec{K}}, \tag{4}$$

where $\{\theta_i\}$ and $\{\theta_i^*\}$ are two independent sets of Grassmann numbers and A is a c-number skew-symmetric $N \times N$ matrix. Notice that the determinant is in the numerator, whereas for c-number Gaussian integrals

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} x^T A x\right) d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}} \, . \tag{5}$$

List of conventions

$$\int d\theta \, \theta = 1 \qquad \text{(normalisation)}$$

$$f(\theta + d\theta) = f(\theta) + d\theta f'(\theta) \qquad \text{(left derivative)}$$

$$d^N\theta = d\theta_N d\theta_{N-1} ... d\theta_2 d\theta_1 \qquad \text{(descending order)}$$

$$d^N\theta^* d^N\theta = d\theta_N^* d\theta_N ... d\theta_1^* d\theta_1 \qquad \text{(descending order)}$$

$$f(\theta) = f_0 + \theta_i f_1^i + ... + \theta_1 \theta_2 ... \theta_N f_N \qquad \text{(ascending order)}$$

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Problem 2 *Scalar QED*

For scalar QED with Lagrangian

$$\mathcal{L}_{sQED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \tag{6}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative:

a) Show that the Lagrangian is invariant under a gauge transformation

$$\phi(x) \to e^{ie\omega(x)}\phi(x), \tag{7}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\omega(x). \tag{8}$$

b) Explain why adding the term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ fixes a gauge.

c) Derive the Feynman rules.

d) Derive the Ward-Takahashi identity in terms of the generating functional for the connected Green functions.

e) Derive the following consequences of the Ward-Takahashi identity

$$(p_\mu - p'_\mu)\,\Gamma^\mu(p,p') = i\left(\frac{1}{S(p)} - \frac{1}{S(p')}\right), \tag{9}$$

$$k_\mu\Gamma^{\mu\nu}(k,p,p') = \Gamma^\mu(p+k,p') - \Gamma^\mu(p,p'-k), \tag{10}$$

where Γ^μ is a vertex with one photon and two scalars, and $\Gamma^{\mu\nu}$ is a vertex with two photons and two scalars. Notice that exactly as in spinor QED, the relation

$$k_\mu D^{\mu\nu}(k) = -i\xi\frac{k^\nu}{k^2} \tag{11}$$

fixes the longitudinal part of the photon propagator $D^{\mu\nu}(k)$.

f) Check the above relations at leading order.

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Problem 3 *Gamma matrices manipulation skills*

a) Using $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ show that in d dimensions

$$\gamma^\mu \gamma_\mu = d\, I_4, \tag{12}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\gamma^\nu, \tag{13}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho} I_4 - (4-d)\gamma^\nu \gamma^\rho, \tag{14}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma. \tag{15}$$

b) For the plain wave solutions to the Dirac equation

$$(\gamma^\mu p_\mu - m)\, u(p,s) = 0, \tag{16}$$

$$(\gamma^\mu p_\mu + m)\, v(p,s) = 0 \tag{17}$$

evaluate the spin sums

$$\sum_{\text{spins}} u(p,s)\bar{u}(p,s) = \gamma^\mu p_\mu + m, \tag{18}$$

$$\sum_{\text{spins}} v(p,s)\bar{v}(p,s) = \gamma^\mu p_\mu - m. \tag{19}$$

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Problem 4 *Anomalous magnetic moment of the electron*

a) Draw the Feynman diagram that represents the leading order contribution to the interaction between a photon and an electron. Write down the amplitude for this diagram.

b) Evaluating $\bar{u}(p',s')\left(\not{p}'\gamma^\mu + \gamma^\mu \not{p}\right)u(p,s)$ in two different ways, prove the Gordon identity

$$\bar{u}(p',s')\gamma^\mu u(p,s) = \bar{u}(p',s')\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p,s). \tag{20}$$

c) Prove that the most general matrix element of the electromagnetic current allowed by Lorentz invariance and current conservation is

$$\begin{aligned}\langle p',s'|J^\mu(x)|p,s\rangle &= \bar{u}(p',s')\left[\gamma^\mu F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}F_2\left(q^2\right)\right]u(p,s)\\ &= \bar{u}(p',s')\left[\frac{(p'+p)^\mu}{2m}F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\Big(F_1\left(q^2\right)+F_2\left(q^2\right)\Big)\right]u(p,s),\end{aligned} \tag{21}$$

where $F_{1,2}\left(q^2\right)$ are two arbitrary functions of the momentum transfer $q^\mu = p^\mu - p'^\mu$, called form factors. $|p,s\rangle$ is a one-particle state of an electron with momentum p^μ and spin s. Notice that $\langle p',s'|J^\mu(x)|p,s\rangle$ transforms as a vector under Lorentz transformations, so a good start is to write down all possible vector interactions. Using translation invariance

$$\langle p',s'|J^\mu(x)|p,s\rangle = \langle p',s'|J^\mu(0)|p,s\rangle\, e^{i(p'-p)x} \tag{22}$$

and the conservation of the electromagnetic current, $\partial_\mu J^\mu = 0$, show that

$$(p'-p)_\mu \langle p',s'|J^\mu(0)|p,s\rangle = 0. \tag{23}$$

The coefficient of the first term in eq. (21) is what we observe as electric charge. Therefore, it is natural to absorb $F_1\left(q^2\right)$ into the definition of the electric charge e, i.e. we set $F_1\left(0\right) = 1$ in the on-shell scheme. On the other hand, the second term is related to the magnetic moment of the electron, which gets shifted from the tree-level value by $1 + F_2(0)$, which we will now calculate to one-loop order.

d) Draw the Feynman diagrams that contribute to next-to-leading order to the interaction between a photon and an electron. Explain why only one of these diagrams contributes to $F_2\left(q^2\right)$ and evaluate this diagram. In the calculation, concentrate on those terms that contribute to $F_2\left(q^2\right)$, which are finite and do not need any regulator, and drop the other contributions. Evaluate your result for $F_2\left(0\right)$ numerically, and compare it to the current experimental value for $(g_e - 2)$.

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