YEAR 2: ELECTRICITY AND MAGNETISM

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PROBLEM SET 4

Note to Tutors: This problem set certainly contains more work than can comfortably be tutored in the remainder of MT. However it is *important that this material is thoroughly covered so I strongly recommend that a tutorial is scheduled for the beginning of HT.*

* Standard bookwork, included because you need to understand this material.

A. Frequency Dependence of the Dielectric Constant

1. (a) Use a simple, classical model of a dilute, non-polar gas to show that the complex refractive index n is given by

$$n^2 - 1 = \frac{N_0 e^2}{\epsilon_0 m} \frac{1}{(\omega_0^2 - \omega^2) - i\gamma\omega}$$

and explain the meaning of the symbols.

(b) Sketch the approximate dependence of the real and imaginary part of the refractive index on the angular frequency of the radiation incident on the medium.

(c) Explain why it is that in a dielectric medium: (i) the relative permittivity measured by a d.c. method is greater than unity; (ii) the phase velocity of X-rays is greater than the velocity of light in free space.

B. Electromagnetic Waves in Good Conductors and Skin Depth.

1. (a) Distinguish between a good conductor and a poor conductor. Estimate the crossover frequency between the two regimes for (i) pure water, (ii) copper. (conductivities: copper 6×10^7 $\Omega^{-1}\mathrm{m}^{-1}$, pure water 5×10^{-6} $\Omega^{-1}\mathrm{m}^{-1}$; static relative permittivity of pure water $\epsilon = 80$, at optical frequencies this has dropped to $\epsilon = 1.8$)

(b) How do Maxwell's equations simplify for a good conductor?

(c) Show that in a good conductor medium (of conductivity σ, relative permittivity ϵ, and relative permeability μ) the field components of a plane wave are propagated as $e^{-\kappa z}e^{i(\omega t-kz)}$ where

$$k=\kappa=\sqrt{\frac{\mu\mu_0\sigma\omega}{2}}. \tag{1}$$

(d) What is the skin depth in a good conductor? Find the skin depth for copper in the visible.

(e) Find the (complex) intrinsic impedance of a good conductor.

C. Electromagnetic Waves at Normal Incidence

1.* A plane electromagnetic wave is incident along the normal to a plane interface separating dielectic media 1 and 2 with relative permittivities ϵ_1 and ϵ_2 respectively. Use the appropriate boundary conditions on $\vec{E}$ and $\vec{H}$ to determine the relative amplitudes of the electric vectors of the incident, reflected and transmitted waves, E_i, E_r and E_t. Use Poynting's vector to determine the relation between the incident, reflected and transmitted intensities and check that energy is conserved.

2. Light is normally incident from a medium 1 with impedance Z_1, through a layer of medium 2 of uniform thickness l and impedance Z_2, into medium 3 of impedance Z_3. Obtain an expression for the total reflected intensity of light when the thickness l corresponds to

(a) $\lambda_2/4$,

(b) $\lambda_2/2$,

where λ_2 is the wavelength of the light in medium 2.

Show that in case (a) the condition for zero reflectance corresponds to $Z_1Z_3=Z_2^2$. Comment on the uses of single and multiple layer coatings in the fabrication of optical components.

3. An electromagnetic wave falls normally on the surface of a good conductor. Show the fraction of power transmitted is approximately $4\sqrt{\epsilon_0\omega/2\sigma}$. Estimate the magnitude of this quantity at radio and at optical frequencies.

D. Plasmas

1. (a) Find the velocity and displacement of a free electron in an electric field $E = E_0 e^{i\omega t}$.

(b) Hence show that the conductivity of a neutral plasma with N free electrons per unit volume is $\sigma = -iNq^2/m\omega$ where m is the mass and q is the charge of an electron.

(c) Show that the dispersion relation for a plane wave $\vec{E}(z,t) = \hat{x}E_0 e^{i(\omega t - \tilde{k}z)}$ moving in a conducting medium is

$$\tilde{k}^2 = \mu\mu_0\epsilon\epsilon_0\omega^2 \left(1 - \frac{i\sigma}{\epsilon\epsilon_0\omega}\right).$$

(d) Putting together (c) and (d), and taking $\mu = \epsilon = 1$, show that the refractive index of a plasma can be written

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

and write down an expression for the plasma frequency ω_p.

(e) Show that the product of the phase and group velocities in the plasma is equal to c^2.

(f) Why may the positive ions be ignored in this calculation?

2. An alternative approach to the same dispersion relation is to treat the plasma as a dielectric ($\sigma = 0$) with an induced polarisation. From the displacement of a free electron, derived in question 1a, find the polarisation $\vec{P}$, relative permittivity ϵ and hence the refractive index n of a neutral plasma of free electrons of density N, together with an equal number of ions.

3. The region between the plates of a parallel plate capacitor is evacuated and filled with a uniform space charge of free electrons of density, N, together with the same density of positive ions. An AC potential $V = V_0 e^{i\omega t}$ is applied. Show that the system behaves like a capacitor (in free space) in parallel with an inductor and find the value of the effective inductance.

4. During a re-entry of the space shuttle it was observed that radio communications at frequencies below 10GHz were interrupted. Calculate the electron density in the neighbourhood of the shuttle.

5. Pulsars are stars that have suffered gravitational collapse. They rotate rapidly and emit a narrow beam of radiation. The pulse lengths, at the earth, are $\sim$1ms and the periods are $\sim$1s.

Within a few months of the discovery of pulsars distance estimates were obtained by exploiting the dispersion of the pulses in the interstellar medium, which is ionised hydrogen with an electron density of about 10^5m^{-3}.

(a) Show that for $\omega^2 \gg \omega_p^2$, where ω_p is the plasma frequency of the interstellar medium, the time delay Δt as a function of $f^{-2} - (f + \Delta f)^{-2}$, where f is the pulse frequency, is a straight line whose slope is a measure of the distance to the pulsar.

(b) For the pulsar CP 0328 the delay between signals at 610 and 408MHz was 0.367s; that between signals at 408 and 151MHz was 4.18s. Find the distance to CP 0328. (The fact that plots such as this give straight lines passing through the origin is evidence that the assumption $\omega^2 \gg \omega_p^2$ is correct, and hence that the delay occurs in interstellar space, not inside the pulsar.)

E. Transmission Lines

1.* (a) Derive the 'Telegraph Equations' for a transmission line

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI, \quad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV$$

where the symbols have their usual meanings, defined per unit length of line.

(b) For a loss-free line ($G = R = 0$) show that V and I satisfy a wave equation. What is the wave velocity v?

(c) If $V(x,t) = f(x - vt) + g(x + vt)$ find I and the characteristic impedance of the line.

2. A transmission line consists of

(a) an air-filled coaxial cable of inner radius a and outer radius b.

(b) a pair of cylindrical conductors of radius a and separation d, in air ($a \ll d$).

(c) a cylindrical conductor of radius a at a distance d in air from a conducting plane ($a \ll d$).

In each case derive the capacitance per unit length C and the inductance per unit length L and check that $1/\sqrt{LC} = c$. Explain why you obtain this value.

3. If Z_1 and Z_2 are the input impedances of a given length l of transmission line when terminated by an open or closed circuit respectively, show that $Z_1 Z_2 = Z_0^2$, independent of l, where

Z_0 is the characteristic impedance of the line.

4. Show that a open circuit transmission line of length $\lambda/8$ and a closed circuit line of length $3\lambda/8$ both have input impedances equal to a pure capacitance of magnitude $C = (\omega Z_0)^{-1}$, where Z_0 is the characteristic impedance of the transmission line.

5. A wave travels along a loss-free transmission line of impedance Z_1 which is terminated by a load of impedance Z_2. Show that a fraction

$$\frac{4Z_1\ Re(Z_2)}{\mid Z_1 + Z_2 \mid^2}$$

of the incident power is transmitted into the load.

F. Rectangular Waveguides (off syllabus & not lectured but technologically important)

1. (a) Derive the equations for the longitudinal components of the electric and magnetic fields in a hollow, loss-free waveguide of constant cross section.
(b) Work out the theory of TE modes for a loss-free rectangular waveguide of dimensions $a \times b, \quad a \geq b$. In particular, find the longitudinal magnetic field, the cutoff frequencies, and the phase and group velocities.
(c) For TM modes find the longitudinal electric field and the cutoff frequencies. Show that the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency is

$$\frac{\omega_{min}^{TM}}{\omega_{min}^{TE}} = \left(1 + \frac{a^2}{b^2}\right)^{1/2}.$$

(d) Why can TEM modes not propagate in this (or any other hollow) waveguide?

2. Consider the resonant cavity produced by closing off the two ends of a rectangular wave guide at $z = 0$ and $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by

$$\omega_{lmn} = v\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$$

for integers l, m and n. Find the associated electric and magnetic fields.